

**Introduction to Theoretical  
Mathematics, Fall 2009  
Homework Assignments**

# Homework 1, due Thursday 10th September

## Question 1

For  $n$  a positive integer, let  $s_n = 1 + 3 + 3^2 + 3^3 + \dots + 3^{n-1}$   
(with  $n$  terms in the sum).

Prove by induction that  $s_n = \frac{3^n - 1}{2}$ .

## Question 2

If  $\mathbb{A}$  and  $\mathbb{B}$  are sets, prove that  $\mathbb{B} \subset \mathbb{A}$  if and only if  $\mathbb{A} \cup \mathbb{B} = \mathbb{A}$ .

## Question 3

Let  $\mathbb{A} = \{x \in \mathbb{R} : -1 \leq x \leq 1 \text{ or } x = 2\}$ .

Let  $\mathbb{B} = \{x \in \mathbb{R} : x = 0 \text{ or } 1 \leq x \leq 2\}$ .

Sketch the sets in the Cartesian plane:  $\mathbb{B} \times \mathbb{B}$ ,  $\mathbb{A} \times \mathbb{B}$ ,  $\mathbb{B} \times \mathbb{A}$  and  $\mathbb{B} \times \mathbb{B}$ .

How are the sets  $\mathbb{A} \times \mathbb{B}$  and  $\mathbb{B} \times \mathbb{A}$  related?

## Question 4

In the Cartesian plane, let  $\mathbb{S} = \{(x, y) \in \mathbb{R}^2 : y^2 = 4x\}$ .

- Sketch the set  $\mathbb{S}$  and prove that  $\mathbb{S}$  is not a function.
- Find three different subsets of  $\mathbb{S}$ , each of which is a function, each with domain the closed interval  $[0, 4]$ .  
For each of your three functions determine its range.

## Question 5

For  $x$  real, let  $f(x) = \frac{2x + 1}{x + 1}$ .

Find the domain and range of  $f$ .

Also give a formula for the composition  $f \circ f$ .

Does  $f$  have an inverse function?

Explain your answer.

## Homework 2, due Thursday 17th September

### Question 1

For  $n$  a positive integer, let  $s_n = 1(2) + 2(3) + 3(4) + \cdots + n(n+1)$   
(with  $n$  terms in the sum).

Prove that  $s_n = \frac{1}{3}n(n+1)(n+2)$ , for any positive integer  $n$ .

### Question 2

For  $n$  an integer, define  $n$  to be even if and only if  $n = 2p$ , for some integer  $p$ .

Also define  $n$  to be odd if and only if  $n+1$  is even.

Prove, using IMP, that every integer is either even or odd and not both.

Redo the proof, using POW instead.

### Question 3

Find with proof, the largest positive integer,  $m$  such that  $n^3 - n$  is divisible by  $m$  for all positive integers  $n$ .

### Question 4

Let  $x_1 = 1$ ,  $x_2 = 2$  and for each positive integer  $n$ , let  $x_{n+2} = \frac{1}{2}(x_n + x_{n+1})$ .

Determine  $x_3$ ,  $x_4$  and  $x_5$  and prove that  $1 \leq x_n \leq 2$  for all  $n \in \mathbb{N}$ .

Also prove that the sequence  $\{x_{2n}; n \in \mathbb{N}\}$  is decreasing.

### Question 5

Prove that  $5^n - 4n - 1$  is divisible by 16 for all  $n \in \mathbb{N}$ .

### Question 6

Let  $f(x) = \frac{x}{\sqrt{x^2 + 1}}$ .

Prove that the function  $f$  gives a bijection from  $\mathbb{R}$  to the interval  $(-1, 1)$  and determine the inverse function of  $f$ .

Also determine, with proof, the sets  $f((0, \infty))$  and  $f^{-1}((0, \infty))$ .

## **Homework 3, due Thursday 24th September**

From section 1.2 of Bartle and Sherbert, do questions 10 and 18.

From section 1.3 of Bartle and Sherbert, do questions 7, 11 and 12.

Also write in your own words a detailed proof of one of the relations of IMP, WIMP and POW to each other.

## **Homework 4, due Thursday 1st October**

Do the take-home quiz due in class Tuesday 29th September.

From section 2.1 of Bartle and Sherbert, do questions 1, 3, 4, 7, 9 and 16.

## **Homework 5, due Thursday 8th October**

From section 2.1 of Bartle and Sherbert, do questions 19 and 24.

From section 2.2 of Bartle and Sherbert, do questions 2, 3, 12 and 13.

## **Homework 6, due Thursday 15th October**

Prepare for the exam, Thursday 15th October.

Everything about the reals, countability, induction, maps and sets.

From section 2.2 of Bartle and Sherbert, do questions 14 and 15.

From section 2.3 of Bartle and Sherbert, do questions 1, 2, 4 and 8.

## **Homework 7, due Thursday 29th October**

From section 2.4 of Bartle and Sherbert, do questions 2, 3, 6 and 14.

From section 2.5 of Bartle and Sherbert, do questions 3 and 16.

## **Homework 8, due Thursday 5th November**

From section 2.5 of Bartle and Sherbert, do question 17.

From section 3.1 of Bartle and Sherbert, do questions 1, 3, 5, 7 and 12.

## **Homework 9, due Thursday 12th November**

From section 3.1 of Bartle and Sherbert, do question 10.

From section 3.2 of Bartle and Sherbert, do questions 1, 2, 6, 9 and 10.

## **Homework 10, due Thursday 19th November**

From section 3.2 of Bartle and Sherbert, do questions 17 and 19.

From section 3.3 of Bartle and Sherbert, do questions 1, 2, 4 and 9.

## **Homework 11, due Thursday 3rd December**

From section 3.3 of Bartle and Sherbert, do questions 11, 12 and 13.

From section 3.4 of Bartle and Sherbert, do questions 3, 7 and 8.

## **Homework 12, due Thursday 10th December**

From section 3.4 of Bartle and Sherbert, do questions 12 and 15.

From section 3.5 of Bartle and Sherbert, do questions 2, 5, 10 and 12.

## **Homework 13, due Thursday 17th December**

From section 3.6 of Bartle and Sherbert, do questions 8 and 9.

From section 3.7 of Bartle and Sherbert, do questions 6, 7, 12 and 14 .