Theoretical Mathematics Exam 2 11/24/9

Name:

Twenty points per question. The best five questions will count, with up to ten bonus points for a sixth question.

Question 1

Let $x$ be the binary number $x = 0.11010010$.
Let $y$ be the base three number $y = 0.11010010$.
(The digits under the bars of these numbers repeat forever).

- Write $x$ and $y$ as fractions.
- Find the infinite decimal expansion of $x$.
- Find the infinite base three expansion of $\frac{3}{y}$.  

Question 2

Let \( S = \left\{ \frac{1}{m^2 + n^2} : m \in \mathbb{N}, n \in \mathbb{N} \right\}. \)

Find with proof the supremum and infimum of the set \( S. \)

Is the set \( S \) closed or open, or neither?

Explain your answer.
Question 3

For any $n \in \mathbb{N}$, let $\mathbb{I}_n = \left[ \frac{-(n + 2)}{n + 1}, \frac{1}{4^n} \right].$

- Prove that $\{\mathbb{I}_n : n \in \mathbb{N}\}$ is a nested sequence of intervals.

- Determine, with proof, the sets $A = \bigcap_{n=1}^{\infty} \mathbb{I}_n$ and $B = \bigcup_{n=1}^{\infty} \mathbb{I}_n.$
Question 4

Let \( x_n = \frac{n^2 + n}{n^2 + 1} \), for any positive integer \( n \).

- Prove, from first principles, that \( \lim_{n \to \infty} x_n = 1 \).
- Also find \( N \), such that \( x_n \) is within \( 10^{-10} \) of 1, for all integers \( n > N \).
Question 5

Determine, with proofs, the following limits, or prove that the limit in question does not exist:

- \[ \lim_{n \to \infty} \left( \frac{n^2}{n^2 + n + 1} \right) \]

- \[ \lim_{n \to \infty} \left( (-1)^n + \frac{(-2)^n}{3n + 1} \right) \]

- \[ \lim_{n \to \infty} \left( \frac{\sqrt{n^3 + 1}}{n + 1} \right) \]
Question 6

For each \( n \in \mathbb{N} \), let \( x_n = \frac{n^2 + 5}{n^2 - 5} \).

- Given \( \epsilon > 0 \), find \( N(\epsilon) \) such that for all \( n > N(\epsilon) \), we have \( |x_n - 1| < \epsilon \).

- Hence find, with proof, each of the following limits (if the limit of the sequence does not exist, determine whether or not the sequence diverges to \( \infty \), or to \( -\infty \)):

  (i) \( A = \lim_{n \to \infty} \left( \sqrt{x_n} - \frac{1}{x_n^2} \right) \)

(ii) \( B = \lim_{n \to \infty} \left( (-1)^n (1 - x_n) \right) \)

(iii) \( C = \lim_{n \to \infty} \left( n \sqrt{n^2 + x_n} - n^2 \right) \)
Question 7

Decide, with proof, which of the following sequences converges.
If a sequence converges, determine its limit.
If a sequence diverges, determine whether or not it diverges to $\infty$, or to $-\infty$.

- $a_n = \frac{(n + 2)^2}{n - 2} - \frac{(n - 2)^2}{n + 2}, \ 3 \leq n \in \mathbb{N}.$
- $b_n = \sqrt{n^4 + n^2} - \sqrt{n^4 - n^2}, \ n \in \mathbb{N},$
- $c_n = \left(1 + \frac{1}{n + 1}\right)^{5n}, \ n \in \mathbb{N},$
- $d_n = \left(1 + \frac{1}{n}\right)^{n^3}, \ n \in \mathbb{N}.$
Question 8

Let \( f(x) \), defined for all real \( x > 0 \), be given by the formula:

\[
f(x) = \frac{2x + 3}{x + 4}.
\]

Consider the recursion \( x_{n+1} = f(x_n) \), with \( x_1 = 1 \).

- Find \( x_2, x_3 \) and \( x_4 \).
- Prove that the sequence \( X = \{x_n : n \in \mathbb{N}\} \) is monotonic.
- Prove that the sequence \( X \) is bounded.
- Prove that \( X \) has a limit \( x \) and determine \( x \).

Also discuss, with proof, the behavior of the sequence generated by the recursion \( x_{n+1} = f(x_n) \), from the initial value \( x_1 = 4 \).