Complex Variables: Quiz 3 Solutions 7/16/9

Question 1

Let $\alpha = (2z(z^2 - 2i) + 2i\overline{z})dz$.
Find the integral of α from $A = (1,1)$ to $B = (4,2)$ along two contours:

- Along the parabola $x = y^2$.
- Horizontally from A to the point $C = (4,1)$ and then vertically to B.

Explain the difference between these two results, using Green's Theorem.

Write $\alpha = \gamma + 2i\overline{z}dz$,

$$\gamma = 2z(z^2 - 2i)dz = df,$$

$$f = \frac{1}{2}(z^2 - 2i)^2.$$

We have:

$$f(A) = f(1+i) = \frac{1}{2}((1+i)^2 - 2i)^2 = \frac{1}{2}(2i - 2i)^2 = 0,$$

$$f(B) = f(4+2i) = \frac{1}{2}((4+2i)^2 - 2i)^2 = \frac{1}{2}(12+14i)^2 = 2(6+7i)^2 = 2(-13+84i) = -26+168i.$$

Then the integral of γ from A to B is independent of the path chosen, so is:

$$\int_A^B \gamma = f(B) - f(A) = -26 + 168i.$$

For the integral of $2i\overline{z}dz$:

- Along the parabola $x = y^2$, we take y as the variable.
  Then we have: $z = x + iy = y^2 + iy$, $\overline{z} = y^2 - iy$, $dz = (2y + i)dy$.
  So we get:

$$2i\overline{z}dz = 2i(y^2 - iy)(2y + i)dy$$

$$= 2i(2y^3 - iy^2 + y)dy = \frac{i}{3}(12y^3 - 6iy^2 + 6y)dy$$

$$= \frac{i}{3}d(3y^4 - 2iy^3 + 3y^2).$$

So the required integral is:

$$-26 + 168i + 2i\int_1^2 (2y^3 - iy^2 + y)dy$$

$$-26 + 168i + \frac{i}{3}[3y^4 - 2iy^3 + 3y^2]_1^2$$

$$-26 + 168i + \frac{i}{3}(3(15) - 14i + 9)$$

$$-26 + 168i + \frac{14}{3} + 18i$$

$$= -\frac{64}{3} + 186i.$$

- Along the line AC, $y = 1$ and we take x as the variable.
  Then we have: $z = x + i$, $\overline{z} = x - i$, $dz = dx$.
  So we get:
  $$2i\overline{z}dz = 2i(x - i)dx$$
  $$= d(ix^2 + 2x).$$
- Along the line CB, $x = 4$ and we take y as the variable.
  Then we have: $z = 4 + iy$, $\overline{z} = 4 - iy$, $dz = idy$.
  So we get:
  $$2i\overline{z}dz = 2i(4 - iy)(idy)$$
  $$= d(iy^2 - 8y).$$

So the required integral is:

$$-26 + 168i + [ix^2 + 2x]_{x=1}^4 + [iy^2 - 8y]_1^2$$

$$= -26 + 168i + 15i + 6 + 3i - 8$$

$$= -28 + 186i.$$

By Cauchy's Theorem, since the differential $\gamma = 2z(z^2 - 2i)dz$ is analytic and globally exact, the difference of the second integral minus the first integral is the integral around the closed loop ACB going first along the lines AC and CB and then back from B to A, along the parabola, of the differential:

$$2i\overline{z}dz = 2i(x - iy)(dx + idy)$$

$$= 2i(xdx + ydy + i(xdy - ydx))$$
$$= d(i(x^2 + y^2)) - 2(xdy - ydx) = -4\Delta.$$

Here Δ is the area of the region with boundary ACB, traced counter-clockwise and we have used Green's Theorem in the last step.

But this area is the area under the parabola and above the line $y = 1$, for $1 \leq x \leq 4$, so is:

$$\Delta = \int_1^4 (\sqrt{x} - 1)dx = \frac{1}{3}\left[2x^{\frac{3}{2}} - 3x\right]_1^4 = \frac{1}{3}[2(7) - 3(3)] = \frac{5}{3}.$$

So the difference of the integrals should be $-4\Delta = -\dfrac{20}{3}$, in agreement with the subtraction:

$$-28 + 186i - \left(-\frac{64}{3} + 186i\right) = \frac{1}{3}(-84 + 64) = -\frac{20}{3}.$$

Question 2

Let $\beta = \dfrac{dz}{z^2(z-1-i)}$.

Find the contour integral of β taken around the following contours all traced once around counter-clockwise:

- $\mathcal{A}$: the circle $|z+1| = 2$
- $\mathcal{B}$: the circle $|z| = 1$
- $\mathcal{C}$: the square $PQRS$ with vertices:

$$P = \left(\frac{1}{2}, \frac{1}{2}\right), \quad Q = \left(\frac{3}{2}, \frac{1}{2}\right), \quad R = \left(\frac{3}{2}, \frac{3}{2}\right), \quad S = \left(\frac{1}{2}, \frac{3}{2}\right).$$

Is the differential β exact on the region outside the circle $|z| = 10$? Explain.

When $z = 1+i$, we have $|z| = \sqrt{2} > 1$ and $|z+1| = \sqrt{5} > 2$, so the pole at $z = 1+i$ of β is outside the contours $\mathcal{A}$ and $\mathcal{B}$, so the integrals for these contours must give the same result and by Cauchy's theorem applied to the double pole at the origin, since the winding number is one, the required integral is:

$$\begin{aligned}
&2\pi i \frac{d}{dz}(z-1-i)^{-1}|_{z=0} \\
&= -2\pi i(z-1-i)^{-2}|_{z=0} \\
&= -2\pi i(-1-i)^{-2} \\
&= -\frac{2\pi i}{(1+i)^2} \\
&= -\frac{2\pi i}{2i} = -\pi.
\end{aligned}$$

The contour $\mathcal{C}$ surrrounds only the pole at $z = 1+i$, so by Cauchy's theorem applied to the simple pole at $z = 1+i$, since the winding number is one, the required integral is:

$$2\pi i z^{-2}|_{z=1+i} = \frac{2\pi i}{(1+i)^2} = \frac{2\pi i}{2i} = \pi.$$

For the last part, put $z = u^{-1}$, $dz = -u^{-2}du$.
Then we have:

$$\begin{aligned}\beta &= \frac{dz}{z^2(z-1-i)} \\ &= \frac{-u^{-2}du}{u^{-2}(u^{-1}-1-i)} \\ &= \frac{udu}{u(1+i)-1} \\ &= \frac{(1-i)udu}{2u-(1-i)} \\ &= \frac{(1-i)udu}{2\left(u-\frac{1}{2}(1-i)\right)}.\end{aligned}$$

In the u-variable, β has only one pole, a simple pole, at the point $u = \dfrac{1}{2}(1-i)$.

In particular, the differential β has no pole at $u = 0$, so no pole at infinity in z-variable, so β is globally exact on any region inside the circle $|u| < \left|\dfrac{1}{2}(1-i)\right| = \dfrac{1}{\sqrt{2}}$, so outside the circle $|z| = \sqrt{2}$.
In particular, the differential β is exact on the region outside the circle $|z| = 10$ and we are done.

Note that the last result explains why the sum of the integrals around $\mathcal{A}$ and $\mathcal{C}$ is zero, since the sum of these contours may be deformed into a large circle surrounding all the poles.

Question 3

Let $\mathcal{T}$ be the transformation $\mathcal{T}(z) = \dfrac{1}{z-1}$ defined for any complex $z \neq 0$.

- Prove that $\mathcal{T}$ is invertible and find a formula for its inverse.

  We solve the equation $z = \mathcal{T}(w)$, for w in terms of z to find the inverse transformation:

  $$z = \frac{1}{w-1}, \quad w - 1 = \frac{1}{z}, \quad w = 1 + \frac{1}{z} = \frac{z+1}{z}.$$

  So we get:

  $$\mathcal{T}^{-1} = 1 + \frac{1}{z} = \frac{1+z}{z}, \text{ defined for any complex } z \neq 0.$$

  Note that $\mathcal{T} : \mathbb{C} - \{1\} \to \mathbb{C} - \{0\}$, since $\mathcal{T}$ is defined provided $z \neq 1$ and since a reciprocal can never be zero, so an output for $\mathcal{T}$ is never zero.

  Also an output for $\mathcal{T}^{-1}$ is never 1, since, if $1 + \dfrac{1}{z} = 1$, we would have $\dfrac{1}{z} = 0$, which is impossible.

  So the map $\mathcal{T}^{-1}$ maps $\mathbb{C} - \{0\} \to \mathbb{C} - \{1\}$, as needed for the inverse of $\mathcal{T}$.
  Then the compositions $\mathcal{T} \circ \mathcal{T}^{-1}$ and $\mathcal{T}^{-1} \circ \mathcal{T}$ are well defined.
  Computing them, we have:

  $$(\mathcal{T} \circ \mathcal{T}^{-1})(z) = \mathcal{T}\left(1 + \frac{1}{z}\right) = \frac{1}{\left(1 + \frac{1}{z} - 1\right)} = \frac{1}{\left(\frac{1}{z}\right)} = z,$$

  $$(\mathcal{T}^{-1} \circ \mathcal{T})(z) = \mathcal{T}^{-1}\left(\frac{1}{z-1}\right) = 1 + \frac{1}{\left(\frac{1}{z-1}\right)} = 1 + z - 1 = z.$$

  We have proved that $\mathcal{T}^{-1}$ is the inverse of $\mathcal{T}$, as required.

  Note that if we allow z to go to infinity, so work on the Riemann sphere, instead of just on $\mathbb{C}$, we have that $\mathcal{T}(\infty) = 0$ and $\mathcal{T}^{-1}(0) = \infty$, whereas $\mathcal{T}(1) = \infty$ and $\mathcal{T}^{-1}(\infty) = 1$, so on the full Riemann sphere, $\mathcal{T}$ extends to give a bijection of the sphere to itself, as does its inverse $\mathcal{T}^{-1}$.

- Find and sketch the images under $\mathcal{T}$ of the following sets:
  - The x-axis.

    We have $z = t$ with t real , so the image of the x-axis is the parametric curve:
$$z = \mathcal{T}(t) = \frac{1}{t-1}.$$
    Note that z is still real, so the image is all the x-axis, except the origin (which is the image of the point at infinity).

    Alternatively if z is in the image, then $\mathcal{T}^{-1}(z)$ lies on the real axis, so we need:
$$\mathcal{T}^{-1}(z) = \overline{\mathcal{T}^{-1}(z)},$$
$$1 + \frac{1}{z} = 1 + \frac{1}{\overline{z}}, \quad \frac{1}{z} = \frac{1}{\overline{z}},$$
$$z = \overline{z}.$$
    This is the equation of the x-axis.

    Alternatively, we see that the images of the three points $(0, 1, \infty)$ on the x-axis are the points $(-1, \infty, 0)$, which also all lie on the x-axis, so the image is the x-axis.
  - The y-axis.

    We have $z = it$ with t real , so the image of the y-axis is the parametric curve:
$$x + iy = z = \mathcal{T}(t) = \frac{1}{it-1} = \frac{-1-it}{1+t^2},$$
$$x = -\frac{1}{1+t^2}, \quad y = -\frac{t}{1+t^2},$$
$$\frac{y}{x} = t, \quad -\frac{1}{x} = 1 + t^2 = 1 + \left(\frac{y}{x}\right)^2,$$
$$-x = x^2 + y^2, \quad x^2 + x + y^2 = 0,$$
$$\left(x + \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}.$$

So the required image is the circle through the origin of radius $\frac{1}{2}$ and center $\left(-\frac{1}{2}, 0\right)$. The origin is the image of the point at infinity.

Alternatively if z is in the image, then $\mathcal{T}^{-1}(z)$ lies on the imaginary axis, so we need:

$$0 = \mathcal{T}^{-1}(z) + \overline{\mathcal{T}^{-1}(z)},$$

$$0 = 1 + \frac{1}{z} + 1 + \frac{1}{\overline{z}},$$

$$0 = 2z\overline{z} + z + \overline{z},$$

$$\frac{1}{4} = z\overline{z} + \frac{1}{2}(z + \overline{z}) + \frac{1}{4},$$

$$\frac{1}{4} = \left(z + \frac{1}{2}\right)\left(\overline{z} + \frac{1}{2}\right) = \left|z + \frac{1}{2}\right|^2,$$

$$\left|z + \frac{1}{2}\right| = \frac{1}{2}.$$

As before this is the equation of a circle through the origin of radius $\frac{1}{2}$ and center $z = -\frac{1}{2}$.

Alternatively, the images of the three points $(0, \infty, i)$ are the three points $\left(-1, 0, \frac{1}{i-1} = \frac{-1-i}{2}\right)$.

So we need the circle through the points $(0,0)$, $(-1,0)$ and $\left(-\frac{1}{2}, -\frac{1}{2}\right)$. We see that these three points are equidistant from the point $\left(-\frac{1}{2}, 0\right)$, the distance being $\frac{1}{2}$, so the image circle is the circle through the origin of radius $\frac{1}{2}$ and center $\left(-\frac{1}{2}, 0\right)$, as before.

– The circle, center the origin, of radius 1.

We parametrize the circle as:

$$z = e^{it},\ t \text{ real}.$$

Then the image is parametrized by:

$$z = \frac{1}{e^{it} - 1},$$

$$\frac{1}{z} = e^{it} - 1,$$

$$1 + \frac{1}{z} = e^{it},$$

$$1 = \left|1 + \frac{1}{z}\right|^2,$$

$$|z|^2 = |1+z|^2 = (1+z)(1+\overline{z}) = 1 + z + \overline{z} + |z|^2,$$

$$0 = 1 + z + \overline{z} = 1 + 2x.$$

So the image is the horizontal line $x = -\dfrac{1}{2}$.

Alternatively, we have the z lies in the image if $\mathbb{T}^{-1}(z)$ lies on the given circle, so if:

$$\left|\mathcal{T}^{-1}(z)\right| = 1,$$

$$\left|1 + \frac{1}{z}\right| = 1.$$

This gives the same equation as that given above.

Alternatively, the images of the three points $(1, -1, i)$ on the circle are $\left(\infty, -\frac{1}{2}, -\frac{1}{2} - \frac{i}{2}\right)$.
These three points are collinear, lying on the line $x = -\dfrac{1}{2}$, which is therefore the image.

– The circle, center the point $(0,1)$, of radius 1.

We parametrize the circle as:

$$z = i + e^{it},\ t \text{ real}.$$

Then the image is parametrized by:

$$z = \frac{1}{i + e^{it} - 1},$$

$$\frac{1}{z} = e^{it} - 1 + i,$$

$$1 + \frac{1}{z} - i = e^{it},$$

$$1 = \left|1 - i + \frac{1}{z}\right|^2,$$

$$|z|^2 = |(1-i)z+1|^2 = (1+(1-i)z)(1+(1+i)\overline{z}) = 1+z+\overline{z}+i(\overline{z}-z)+2|z|^2,$$

$$1 = 2 + z(1-i) + \overline{z}(1+i) + z\overline{z}$$

$$1 = (z+1+i)(\overline{z}+1-i) = |z+1+i|^2.$$

So the image is the circle of unit radius, center the point $z = -1-i$.

Alternatively, the images of the three points $(0, i+1, 2i)$ on the circle are $\left(-1, -i, \frac{-1-2i}{5}\right)$.

We see that these three points $(-1,0)$, $(0,-1)$ and $\left(-\frac{1}{5}, -\frac{2}{5}\right)$ are equidistant from the point $(-1,-1)$, the distance being 1, so the image circle is the circle through the origin of radius 1 and center $(-1,-1)$, as before.