Complex Variables, Summer 2009
Solutions to non-book problems

Homework 1 Solutions

Question 1

Let $a = 3 + 2i$ and $b = 4 + 3i$.
Sketch the complex numbers a, b, $a + b$, $a - b$, a^2, ab, $a\bar{b}$, b^2, $\frac{a}{b}$ and $\frac{b}{a}$.
For each of these numbers, determine their modulus and argument.

$$a = 3 + 2i, \quad |a| = \sqrt{9+4} = \sqrt{13} = 3.60555,$$

$$\arg(a) = \arctan\left(\frac{2}{3}\right) = 0.588003 \text{ radians } = 33.6901 \text{ degrees},$$

$$b = 4 + 3i, \quad |b| = \sqrt{16+9} = \sqrt{25} = 5,$$

$$\arg(b) = \arctan\left(\frac{3}{4}\right) = 0.643501 \text{ radians } = 36.8699 \text{ degrees},$$

$$a + b = 7 + 5i, \quad |a+b| = \sqrt{49+25} = \sqrt{74} = 8.60233,$$

$$\arg(a+b) = \arctan\left(\frac{5}{7}\right) = 0.620249 \text{ radians } = 35.5377 \text{ degrees},$$

$$a - b = -1 - i, \quad |a-b| = \sqrt{1+1} = \sqrt{2} = 1.414214,$$

$$\arg(a-b) = \pi + \arctan(1) = \frac{5\pi}{4} = 3.926991 \text{ radians } = 225 \text{ degrees},$$

$$a^2 = (3+2i)^2 = 5 + 12i, \quad |a^2| = |a|^2 = \left(\sqrt{13}\right)^2 = 13 = \sqrt{169} = \sqrt{25+144},$$

$$\arg(a^2) = 2\arg(a) = \arctan\left(\frac{12}{5}\right) = 2\arctan\left(\frac{2}{3}\right) = 1.176005 \text{ radians } = 67.3801 \text{ degrees},$$

$$ab = (3{+}2i)(4{+}3i) = 6{+}17i, \quad |ab| = |a||b| = 5\sqrt{13} = \sqrt{325} = \sqrt{36+289} = 18.0278,$$

$$\arg(ab) = \arg(a){+}\arg(b) = \arctan\left(\frac{17}{6}\right) = 1.2315 \text{ radians } = 70.5600 \text{ degrees},$$

$$a\bar{b} = (3{+}2i)(4{-}3i) = 18{-}i, \quad |a\bar{b}| = |a||b| = 5\sqrt{13} = \sqrt{325} = \sqrt{1+324} = 18.0278,$$

$$\arg(a\bar{b}) = \arg(a){-}\arg(b) = -\arctan\left(\frac{1}{18}\right) = -0.055499 \text{ radians } = -3.1798 \text{ degrees},$$

$$b^2 = (4+3i)^2 = 7 + 24i, \quad |b^2| = |b|^2 = 5^2 = 25 = \sqrt{625} = \sqrt{49+576},$$

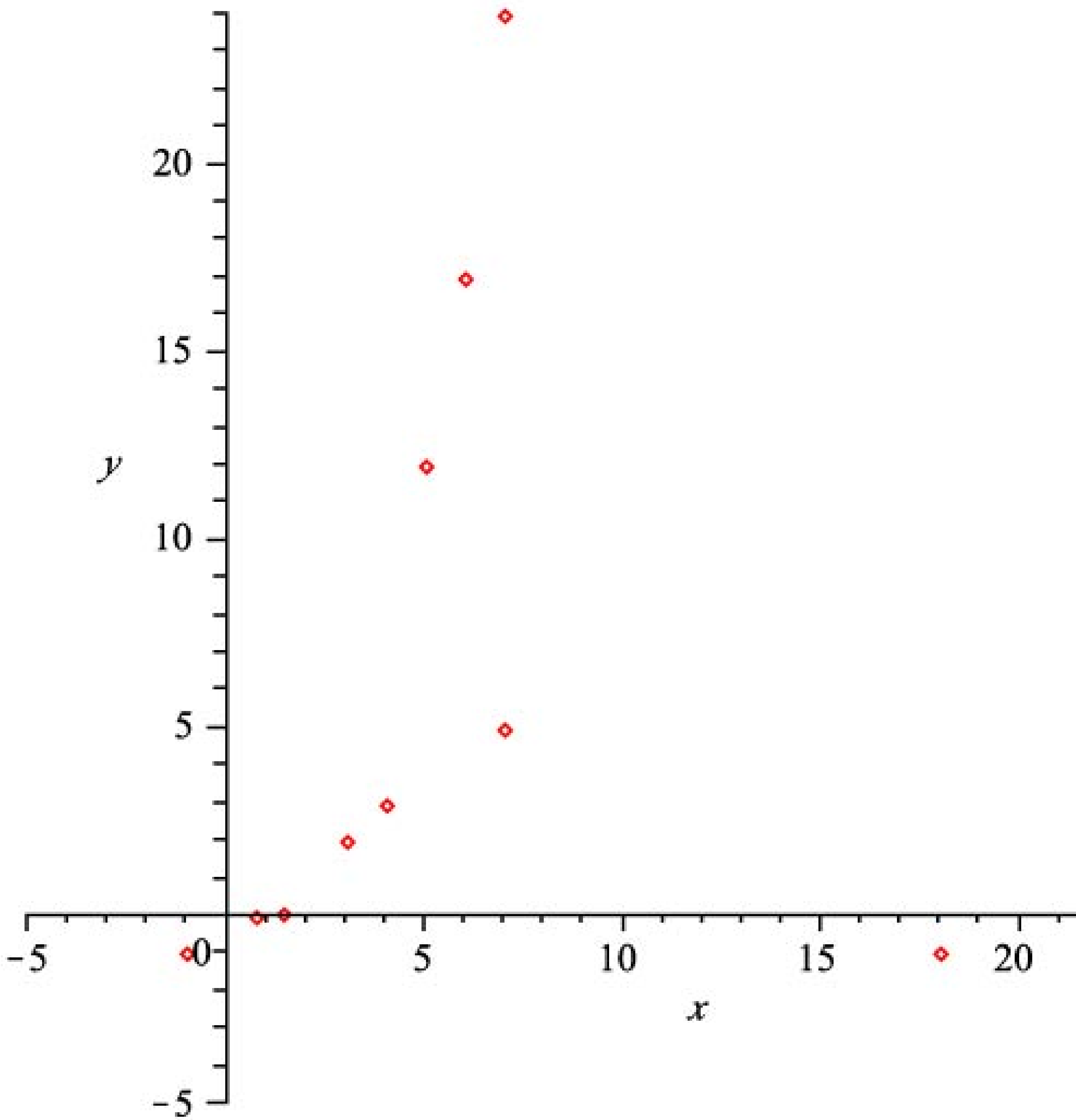
$$\arg(b^2) = 2\arg(b) = \arctan\left(\frac{24}{7}\right) = 2\arctan\left(\frac{4}{3}\right) = 1.2870 \text{ radians } = 73.7398 \text{ degrees},$$

$$\frac{a}{b} = \frac{a\bar{b}}{|b|^2} = \frac{1}{25}(18-i), \quad \left|\frac{a}{b}\right| = \frac{5\sqrt{13}}{25} = \frac{\sqrt{13}}{5} = 0.721110,$$

$$\arg\left(\frac{a}{b}\right) = \arg(a\bar{b}) = \arg(a) - \arg(b) = -\arctan\left(\frac{1}{18}\right) = -0.055499 \text{ radians } = -3.1798 \text{ degrees},$$

$$\frac{b}{a} = \frac{b\bar{a}}{|a|^2} = \frac{\overline{(a\bar{b})}}{|a|^2} = \frac{1}{13}(18+i), \quad \left|\frac{a}{b}\right| = \frac{5\sqrt{13}}{13} = \frac{\sqrt{13}}{5} = 1.386750,$$

$$\arg\left(\frac{b}{a}\right) = \arg(b\bar{a}) = \arg(b) - \arg(a) = \arctan\left(\frac{1}{18}\right) = 0.055499 \text{ radians } = 3.1798 \text{ degrees}.$$



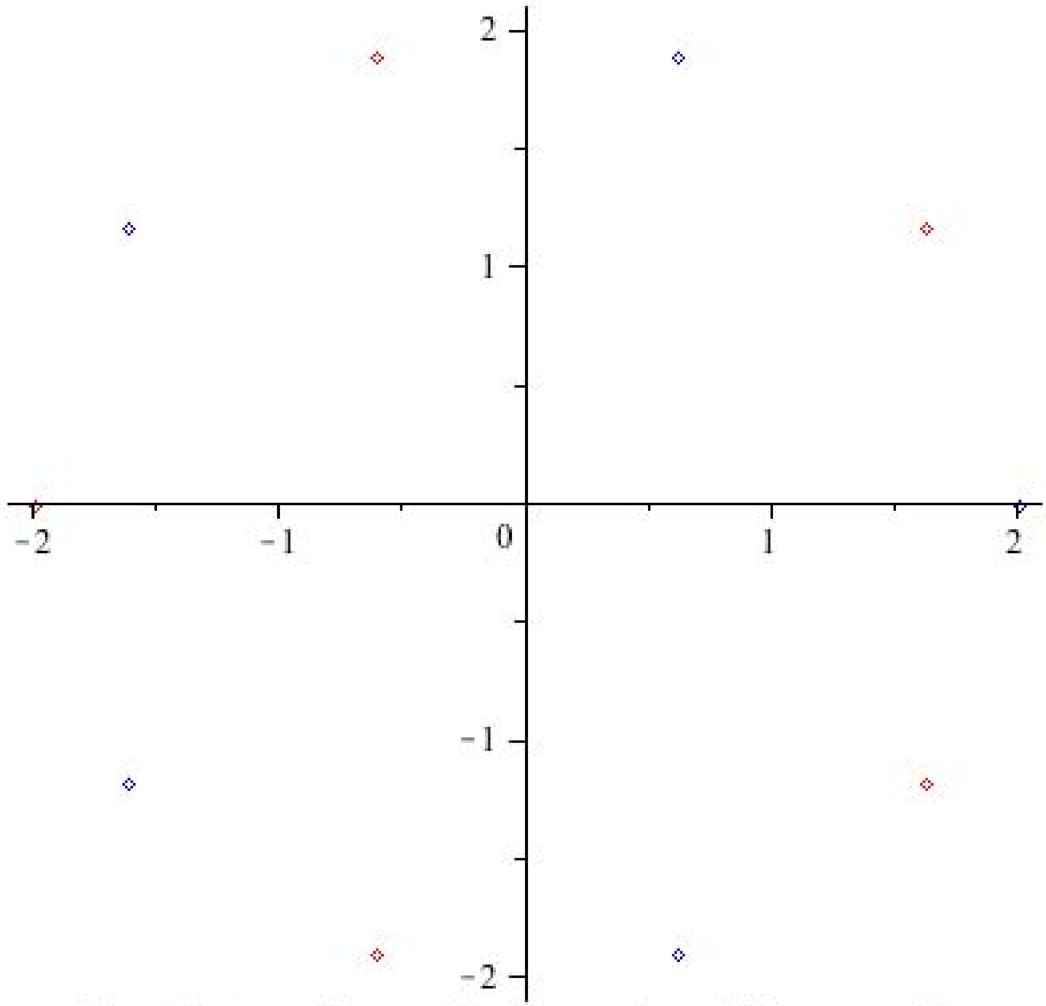
The ten points plotted on the complex plane

Question 2

Find the solutions of the equation $z^5 = -32$ and all solutions of the equation $z^{10} = 1024$ and plot the solutions in the complex plane.

If $z^5 = -32$, then $z^{10} = (-32)^2 = 1024$, so if we solve the second equation, we can infer the solutions of the first.

For the equation $z^{10} = 1024 = 2^{10}$, put $z = 2w$, then $w^{10} = 1$, so the solutions are $w = e^{\frac{2ik\pi}{10}} = e^{\frac{ik\pi}{5}}$, giving $z = 2e^{\frac{ik\pi}{5}}$ as the general solution of the equation $z^{10} = 1024$, where k is an integer.
Plotting, the roots form the vertices of a regular decagon, inscribed in the circle center the origin and radius 2.
Starting at the point 2, we get the roots at intervals of $\frac{\pi}{5}$ radians, or 36 degrees, around the circle.
Note that we then have $z^5 = 32e^{ik\pi} = -32$, provided k is odd, so the roots of $z^5 = -32$ form a regular pentagon inscribed inside the decagon, beginning at the point $z = -2$, the other roots occurring at intervals of $\frac{2\pi}{5}$ radians, or 72 degrees around the circle.



The red points of the regular decagon have fifth power -32

Question 3

A complex number z obeys the relations $|z+i| = 3\sqrt{2}$ and $|z-3| = 4$.
What can we say about the number z?
In particular is z unique? Explain your answer graphically.

We have z lying on the intersection of two distinct circles, so there are at most two such z.
The first circle is:

$$18 = |z+i|^2 = |x+i(y+1)|^2 = x^2 + (y+1)^2.$$

The second is:

$$16 = |z-3|^2 = |x-3+iy|^2 = (x-3)^2 + y^2.$$

Subtracting, we get:

$$-2 = (x-3)^2 + y^2 - x^2 - (y+1)^2 = 8 - 6x - 2y,$$

$$y = 5 - 3x.$$

Back substituting, we get:

$$16 = (x-3)^2 + (5-3x)^2 = 10x^2 - 36x + 34,$$

$$5x^2 - 18x + 9 = 0.$$

$$(5x-3)(x-3) = 0,$$

$$(x,y) = (3,-4), \quad (x,y) = \frac{1}{5}(3,16),$$

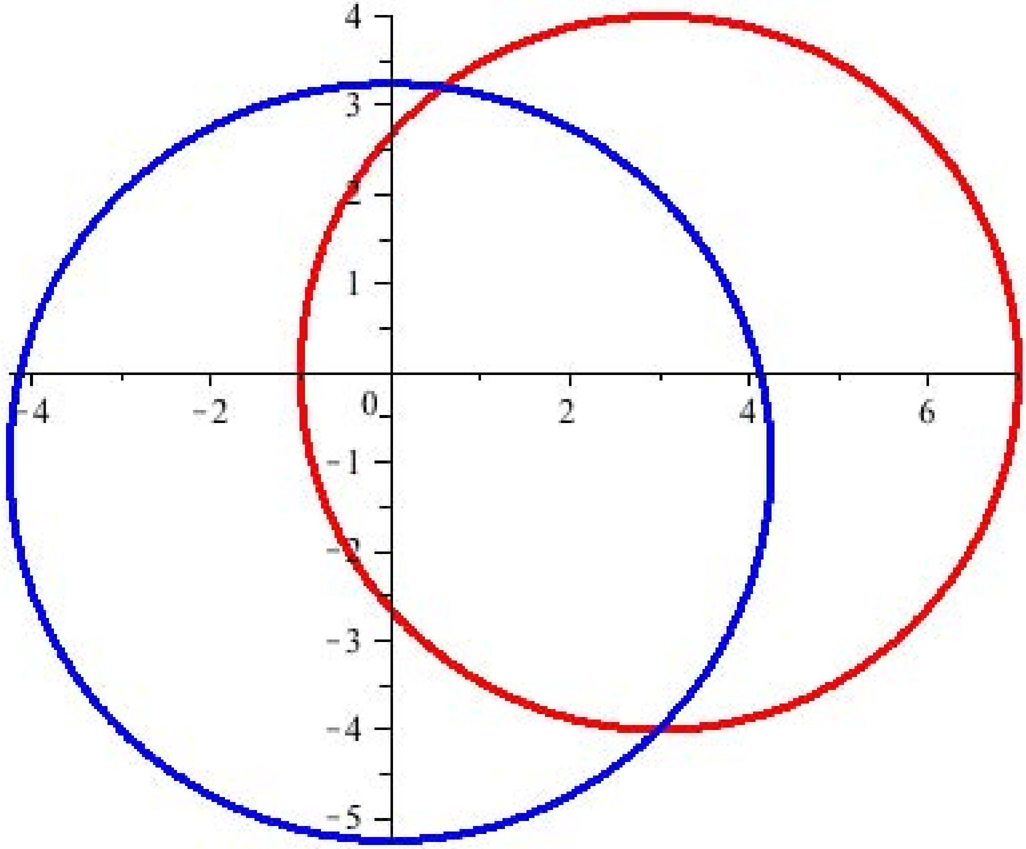
$$z = 3-4i, \quad z = \frac{1}{5}(3+16i).$$

Check:
With $z = 3-4i$, we have:

$$|z+i| = |3-3i| = 3\sqrt{2}, \quad |z-3| = |-4i| = 4.$$

With $z = \frac{1}{5}(3+16i)$, we have:

$$|z+i| = \frac{3}{5}|1+7i| = \frac{3}{5}\sqrt{50} = 3\sqrt{2}, \quad |z-3| = \frac{4}{5}|-3+4i| = \frac{4}{5}\sqrt{25} = 4.$$



The circles meet at two points

Question 4

Solve (algebraically) the equations $z^2 = -3+4i$ and $z^2 = -3-4i$ and plot the solutions on the complex plane.

Clearly if we conjugate the first equation and replace $\overline{z}$ by z, we get the second and vice-versa, so we only need solve the first equation and conjugate to get the solutions to the second.

For the first equation, if $z^2 = -3+4i$, then we have:

$$\left|z^2\right| = |z|^2 = |-3+4i| = \sqrt{9+16} = \sqrt{25} = 5.$$

Put $z = x+iy$, with x and y real.
Then we have:

$$x^2+y^2 = |z|^2 = 5,$$
$$x^2-y^2+2ixy = z^2 = -3+4i,$$
$$xy = 2, x^2-y^2 = -3.$$

Then $2x^2 = (x^2-y^2)+(x^2+y^2) = -3+5 = 2$, so $x^2 = 1$ and $x = \pm 1$, so $(x,y) = (1,2)$ or $(x,y) = (-1,-2)$.
So the solutions are $z = 1+2i$ and $z = -(1+2i)$.
Check: if $z = \pm(1+2i)$, then $z^2 = 1-4+4i = -3+4i$, as required.
Then the solutions of $z^2 = -3-4i$ are $z = \pm(1-2i)$.
Plotting, the four points are the points $(\pm 1, \pm 2)$, which form a 2×4 rectangle symmetric about the axes.
The solutions of the equation $z^2 = -3+4i$ are the vertices of the rectangle in the first and third quadrants.
The solutions of the equation $z^2 = -3-4i$ are the vertices of the rectangle in the second and fourth quadrants.