Complex variables: Exam 1 Solutions 7/9/9

Question 1

Determine the following limits, or explain why the limit in question does not exist.

• \[ \lim_{z \to 1+i} \left( \frac{z^4 + 2iz^2 + 8}{z^2 - 3iz - 3 + i} \right) \]

When \( z = 1 + i \), we have \( z^2 = 1 - 1 + 2i = 2i \), so \( z^4 = -4 \), so \( z^4 + 2iz^2 + 8 = -4 + 2i(2i) + 8 = 0 \) and \( z^2 - 3iz - 3 + i = 2i - 3i(1 + i) - 3 + i = 0 \).

So we may use L'Hôpital's rule, giving:

\[
\lim_{z \to 1+i} \left( \frac{z^4 + 2iz^2 + 8}{z^2 - 3iz - 3 + i} \right) = \lim_{z \to 1+i} \left( \frac{4z^3 + 4iz}{2z - 3i} \right) \\
= 4 \lim_{z \to 1+i} \left( \frac{z(z^2 + i)}{2z - 3i} \right) = \frac{12}{5} \left( \frac{i - 1}{2 - i} \right) \\
= \left( \frac{12}{5} \right) (i - 1)(2 + i) = \left( \frac{12}{5} \right) (-3 + i) = -7.2 + 2.4i.
\]

Alternatively, we can factor:

\[
z^4 + 2iz^2 + 8 = (z^2 + 4i)(z^2 - 2i) = (z^2 + 4i)(z - 1 - i)(z + 1 + i),
\]

\[
z^2 - 3iz - 3 + i = (z - 1 - i)(z + 1 - 2i).
\]

\[
\lim_{z \to 1+i} \left( \frac{z^4 + 2iz^2 + 8}{z^2 - 3iz - 3 + i} \right) = \lim_{z \to 1+i} \left( \frac{4z^3 + 4iz}{2z - 3i} \right) \\
= \lim_{z \to 1+i} \left( \frac{(z^2 + 4i)(z - 1 - i)(z + 1 + i)}{(z - 1 - i)(z + 1 - 2i)} \right) \\
= \lim_{z \to 1+i} \left( \frac{(z^2 + 4i)(z + 1 + i)}{(z + 1 - 2i)} \right) \\
= \left( \frac{(2i + 4)(2 + 2i)}{2 - i} \right) = 12 \left( \frac{(i - 1)(2 + i)}{5} \right) = \left( \frac{12}{5} \right) (-3 + i) = -7.2 + 2.4i.
\]
\[
\lim_{z \to 0} \left( \frac{\ln(1 - z^2)}{\sin^2(2z)} \right)
\]

Since \( \sin(0) = \ln(1) = 0 \), this is a standard "0/0" limit, so we may apply L'Hôpital's rule, giving, by the chain rule:

\[
\lim_{z \to 0} \left( \frac{\ln(1 - z^2)}{\sin^2(2z)} \right) = \lim_{z \to 0} \left( \frac{-\frac{2z}{1-z^2}}{4\sin(2z)\cos(2z)} \right)
\]

\[
= \lim_{z \to 0} \left( \frac{-\frac{2z}{1-z^2}}{4\sin(2z)\cos(0)} \right) = -\left( \frac{1}{2} \right) \lim_{z \to 0} \left( \frac{z}{\sin(2z)} \right)
\]

Here we again have a standard "0/0" limit, so we may apply L'Hôpital's rule, giving:

\[
-\left( \frac{1}{2} \right) \lim_{z \to 0} \left( \frac{z}{\sin(2z)} \right) = -\left( \frac{1}{2} \right) \lim_{z \to 0} \left( \frac{1}{2\cos(2z)} \right) = -\left( \frac{1}{2} \right) \left( \frac{1}{2\cos(0)} \right) = -\frac{1}{4}.
\]

So the required limit exists and is \(-\frac{1}{4}\).

\[
\lim_{z \to 0} \left( \frac{\Re(z^2)\Im(z)}{z^3} \right)
\]

Put \( z = re^{i\theta} \), where \( r > 0 \) and \( \theta \) are both real.

Then we have \( z^3 = r^3 e^{3i\theta} \). Also \( \Re(z^2) = \Re(r^2 e^{2i\theta}) = r^2 \cos(2\theta) \) and \( \Im(z) = \Im(re^{i\theta}) = r \sin(\theta) \), so the required limit becomes:

\[
\lim_{z \to 0} \left( \frac{\Re(z^2)\Im(z)}{z^3} \right) = \lim_{r \to 0^+} \left( \frac{r^3 \cos(2\theta) \sin(\theta)}{r^3 e^{3i\theta}} \right) = \lim_{r \to 0^+} \left( e^{-3i\theta} \cos(2\theta) \sin(\theta) \right).
\]

If we approach \( z = 0 \) along the positive \( x \)-axis, we have \( \theta = 0 \), so the limit is \( e^0 \cos(0) \sin(0) = 0 \).

If we approach \( z = 0 \) along the line \( \theta = \frac{\pi}{6} \), the limit is \( e^{\frac{i\pi}{6}} \cos \left( \frac{\pi}{3} \right) \sin \left( \frac{\pi}{6} \right) = -\frac{i}{4} \).

Since these two limits differ, the required limit does not exist.

Alternatively, we could first take \( z \neq 0 \) real, giving limit zero, since \( \Im(z) = 0 \).

Then we take \( z = ti \), where \( t \neq 0 \) is real, so we have \( z^3 = -it^3 \), \( \Im(z) = t \)

and \( \Re(z^2) = \Re(-t^2) = -t^2 \), giving the limit \( \lim_{t \to 0} \frac{-t^2}{-it^3} = -i \).

Again, since these two limits differ, the required limit does not exist.
Question 2

Sketch the following sets in the complex plane and for each identify whether the set is open, closed or neither and whether or not the set is bounded, connected or compact.

For each of these sets also give a parametrization or parametrizations of its boundary, as appropriate, where the boundary is traced counter-clockwise with respect to an observer in the set.

- \( \mathbb{A} = \{ z : 4 \Re(z) \leq \Im(z) - 4 \} \).

Put \( z = x + iy \), with \( x \) and \( y \) real.
Then we need \( 4x \leq y - 4 \), or \( y \geq 4x + 4 \).
So the region \( \mathbb{A} \) is the closed-half-plane lying on or above the straight line \( y = 4x + 4 \), which has slope 4 and inteccepts \((0, 4)\) and \((-1, 0)\).
Then \( \mathbb{A} \) is closed, unbounded, non-compact and connected.
Its boundary is the line \( y = 4x + 4 \), which is parametrized as \( x = t \) and \( y = 4t + 4 \), for any real \( t \).
The line goes up from bottom left to top right.
The region A is the yellow region with the blue line included. The region is closed but unbounded.
• $B = \{ z : |z - 2| \leq 2 \text{ and } |z - 2i| \leq 2 \}.$

The region $B$ is the closed region common to the two circles of radius 2, one centered at $(2, 0)$, the other at $(0, 2)$.

It is symmetrical about the line $y = x$ and is bounded by two circular arcs, which meet at the points $z = 0 = (0, 0)$ and $z = 2 + 2i = (2, 2)$ on that line.

Then $B$ is closed, bounded, connected and compact.
The part of the boundary on the circle $|z-2| = 2$ goes counter-clockwise from $z = 2 + 2i$ to $z = 0$, so is $z = 2 + 2e^{is}$, $\frac{\pi}{2} \leq s \leq \pi$.

The part of the boundary on the circle $|z-2i| = 2$ goes counter-clockwise from $z = 0$ to $z = 2 + 2i$, so is $z = 2i + 2e^{it}$, $-\pi \leq t \leq 0$. 

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\( \mathbb{C} = \left\{ z = r e^{i\theta} : 0 < r \leq 3 \text{ and } -\frac{2\pi}{3} \leq \theta \leq 0 \right\} \).

This is a sector of the disc centered at the origin, of radius 3 units, in the third and fourth quadrants bounded above by the positive \( x \)-axis and by the line through the origin sloping up at 60 degrees, except that the origin is deleted.

Since the origin is deleted, \( \mathbb{C} \) is neither closed nor open, is bounded, non-compact and connected. The boundary of \( \mathbb{C} \) has three pieces:

- The circular arc from \( z = 3e^{-\frac{2\pi}{3}} \) to \( z = 3 \):
  
  \[ z = 3e^{i\theta}, \quad -\frac{2\pi}{3} \leq \theta \leq 0, \]

- The real axis, from \( z = 3 \) to \( z = 0 \)
  
  \[ z = 3 - s, \quad 0 \leq s \leq 3, \]

- The line of slope 60 degrees through the origin from \( z = 0 \) to \( z = 3e^{-\frac{2\pi}{3}} \):
  
  \[ z = te^{-\frac{2\pi}{3}}, \quad 0 \leq t \leq 3. \]

Note that the boundary of \( \mathbb{C} \) includes the origin, even though it is not in \( \mathbb{C} \), because it is a limit point of \( \mathbb{C} \).
The region $C$ is a sector of a disc of radius 3. The origin is not a part of the region.
Question 3

Let \( g(z) = \frac{2z - 1}{z + 2} \) be defined for any complex number \( z \neq -2 \).

- Prove that the function \( g(z) \) is analytic on its domain and compute its derivative from first principles.

  We need to show that the complex derivative of \( g \) exists everywhere, so we need to compute the limit, for \( z \neq -2 \) and \( w \neq -2 \):

  \[
  g'(z) = \lim_{w \to z} \frac{g(w) - g(z)}{w - z} = \lim_{w \to z} \frac{2w-1}{w+2} - \frac{2z-1}{z+2} \]

  \[
  = \lim_{w \to z} \frac{(2w-1)(z+2) - (2z-1)(w+2)}{(w-z)(z+2)(w+2)} = \frac{1}{(z+2)^2} \lim_{w \to z} \frac{5w - 5z}{w - z} = \frac{1}{(z+2)^2} \lim_{w \to z} 5 = \frac{5}{(z+2)^2}.
  \]

  So \( g \) is analytic on its domain, as required.

- Is \( \Re(g) \) harmonic on its domain? Explain.

  Yes every analytic function obeys the Cauchy-Riemann equations and its real and imaginary parts are then harmonic.

- Find, at least locally, a function \( f(z) \) such that \( d(f(z)) = g(z)dz \).

  For \( x \) real, we compute the integral:

  \[
  \int \left( \frac{2x-1}{x+2} \right) dx = \int \left( \frac{2u-5}{u} \right) du = \int \left( 2 - \frac{5}{u} \right) du
  \]

  \[
  = 2u - 5 \ln(u) = 2(x + 2) - 5 \ln(x + 2).
  \]

  Here we put \( x = u - 2, \) \( u = x + 2, \) \( du = dx. \)

  So \( f(z) = 2(z + 2) - 5 \ln(z + 2) \) will do.

- Does the domain of your solution for \( f(z) \) match the domain of the function \( g(z) \)?

  Explain.

  No. The function \( f(z) \) requires a branch cut from \(-2\) to infinity, whereas the function \( g(z) \) does not.
Question 4

Let \( u = x^4 + y^4 - 6x^2y^2 + 2x^2 - 2y^2 + 1 \)

- Prove that \( u \) is harmonic and find its harmonic conjugate \( v \).

We have:

\[
x^2 - y^2 = \Re(z^2),
\]

\[
x^4 + y^4 - 6x^2y^2 = \Re(z^4).
\]

So \( u = \Re(z^4 + 2z^2 + 1) = \Re((z^2 + 1)^2) \).

So \( u \) is harmonic, with harmonic conjugate:

\[
v = \Im(z^4 + 2z^2 + 1) = 4x^3y - 4xy^3 + 4xy + c.
\]

Here \( c \) is an arbitrary real constant.

Alternatively, we solve the Cauchy-Riemann equations:

\[
v_x = -u_y = -4y^3 + 12x^2y + 4y,
\]

\[
v = -4xy^3 + 4x^3y + 4xy + k(y),
\]

\[
0 = v_y - u_x = k'(y) - 12xy^2 + 4x^3 + 4x - 4x^3 + 12xy^2 - 4x = k'(y).
\]

So \( k \) is constant and the harmonic conjugates of \( u \) are the functions \( v = -4xy^3 + 4x^3y + 4xy + k \), where \( k \) is an arbitrary constant.

- Express \( f = u + iv \) as a function of \( z = x + iy \) and show that:

\[
f'(z) = \partial_x f = -i\partial_y f.
\]

Explain why \( f \) is unique, given that \( f(0) = 1 \).

The freedom in choosing the harmonic conjugate \( v \) of an harmonic function \( u \) is just a constant, since the derivatives \( v_x \) and \( v_y \) are known.

The constant is fixed uniquely by specifying the value of the harmonic conjugate at one point.

Here the requirement \( f(0) = 1 \) gives \( v(0) = 0 \), so \( v \) is uniquely determined, as required.
We have \( u + iv = f(z) \), where \( f(z) = z^4 + 2z^2 + 1 = (z^2 + 1)^2 \).

Then we have, by the chain rule:

\[
f'(z) = 2(z^2 + 1)(2z) = 4z(z^2 + 1) = 4(x + iy)(x^2 - y^2 + 1 + 2ixy)
\]

\[
= 4x^3 - 12xy^2 + 4x + iy(12x^2 - 4y^2 + 4)
\]

Also we have:

\[
f_x = u_x + iv_x = 4x^3 - 12xy^2 + 4x + i(12x^2y - 4y^3 + 4y)
\]

\[
= 4x^3 - 12xy^2 + 4x + iy(12x^2 - 4y^2 + 4) = f'(z),
\]

\[-if_y = v_y - iu_y + v_y = 4x^3 - 12xy^2 + 4x - i(4y^3 - 12x^2y - 4y) = f_x = f'(z).
\]

- When \( f(0) = 1 \), factorize the function \( u^2 + v^2 \) as a product of complex functions, each linear in the variables \( x \) and \( y \) (or, equivalently, linear in the variables \( z \) and \( \overline{z} \)).

We have:

\[
u^2 + v^2 = (u + iv)(u - iv) = (z^2 + 1)^2(\overline{z}^2 - 1)^2
\]

\[
= (z + i)(z - i)(\overline{z} + i)(\overline{z} - i)(z - i)(\overline{z} - i)(z + i)(\overline{z} + i)
\]

\[
= (x + i(y + 1))(x + i(y - 1))(x + i(y + 1))(x + i(y - 1))(x + i(y + 1))(x + i(y - 1))
\]

\[
= (x - i(y - 1))(x - i(y - 1))(x - i(y + 1))(x - i(y + 1)).
\]
Question 5

Consider the transformation $T : z \rightarrow (1 + i)z + 3 - 4i$, defined for any $z \in \mathbb{C}$.

- Find the image $A'B'C'D'$ under $T$ of the square $ABCD$ with vertices $A = 1 + i$, $B = -1 + i$, $C = -1 - i$, and $D = 1 - i$.

How do the areas of $ABCD$ and $A'B'C'D'$ compare?

We have:

- $A' = T(A) = (1 + i)(1 + i) + 3 - 4i = 2i + 3 - 4i = 3 - 2i$,
- $B' = T(B) = (1 + i)(-1 + i) + 3 - 4i = -2 + 3 - 4i = 1 - 4i$,
- $C' = T(C) = -(1 + i)(1 + i) + 3 - 4i = -2i + 3 - 4i = 3 - 6i$,
- $D' = T(D) = -(1 + i)(-1 + i) + 3 - 4i = 2 + 3 - 4i = 5 - 4i$.

Plotting we see that $A'B'C'D'$ is a square of side length:

$$|A'B'| = |B' - A'| = |-2 - 2i| = \sqrt{4 + 4} = 2\sqrt{2}.$$  

So the area of $A'B'C'D' = (2\sqrt{2})^2 = 8$ units of area.

This is double the area of $ABCD$ which is a square of side length 2, so has area 4 units of area.

- Find the fixed points of the transformation $T$, if any.

The fixed points are the solutions of the equation:

$$T(z) = z,$$

$$(1 + i)z + 3 - 4i = z,$$

$$iz = 4i - 3,$$

$$z = -i(4i - 3) = 4 + 3i.$$

Check:

$$T(4 + 3i) = (1 + i)(4 + 3i) + 3 - 4i = 4 - 3 + 4i + 3i + 3 - 4i = 4 + 3i.$$  

So the only fixed point is the point $z = 4 + 3i = (4, 3)$. 

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• Describe the transformation $\mathcal{T}$ geometrically.

We see that the transformation is a combination of a counter-clockwise rotation through forty-five degrees and a dilation by a factor of $\sqrt{2}$ centered at the point $(4,3)$.

In fact, since $1 + i = \sqrt{2} e^{\frac{i\pi}{4}}$, we have:

$$
\mathcal{T} = \sqrt{2} e^{\frac{i\pi}{4}} z + 3 - 4i.
$$

The factor of $\sqrt{2}$ gives the dilation.

The factor of $e^{\frac{i\pi}{4}}$ gives the rotation.

The term $3 - 4i$ gives a translation, which has the effect of moving the center of the transformation to the invariant point $(4,3)$.

• Find a formula for the inverse transformation, $\mathcal{T}^{-1}$.

We solve the equation:

$$
w = \mathcal{T}(z) = (1 + i)z + 3 - 4i,
$$

$$(1 - i)w = (1 - i)(1 + i)z + (1 - i)(3 - 4i) = 2z - 1 - 7i,$$

$$
z = \frac{1}{2}((1 - i)w + 1 + 7i).
$$

So the inverse transformation is:

$$
\mathcal{T}^{-1}(z) = \frac{1}{2}((1 - i)z + 1 + 7i).
$$
• Find the image under the transformation $T$ of the circle $|z| = 5$ and sketch the circle and its image on the complex plane. Also give an equation for the image circle.

Since the transformation is a combination of rotations, translations and dilations, it maps straight lines to straight lines and circles to circles and scales all distances by the dilation factor $\sqrt{2}$.

So the image of the given circle which has radius 5 is a circle whose radius is $5\sqrt{2}$.

Since the given circle has center at the origin, the image circle has its center at the image of the origin, which is the point $T(0) = 3 - 4i$.

So the image circle is the circle center $3 - 4i = (3, 4)$ and radius $5\sqrt{2}$. Its equation is $|z - 3 + 4i| = 5\sqrt{2}$, or squaring:

$$50 = (x - 3)^2 + (y + 4)^2,$$
$$x^2 + y^2 - 6x + 8y = 25.$$  

Alternatively, we parametrize the given circle:

$$z = 5e^{it}, \quad t \text{ real}.$$  

Then the image of the given circle under the transformation $T$ is the collection of points:

$$z = T(5e^{it}) = 5(1 + i)e^{it} + 3 - 4i.$$  

Then we have:

$$z - 3 + 4i = 5(1 + i)e^{it}.$$  

Taking the absolute value of both sides, we eliminate $t$, since $|e^{it}| = 1$, for $t$ real.

So we get the equation of the image as:

$$|z - 3 + 4i| = |5(1 + i)e^{it}| = 5|1 + i| = 5\sqrt{2}.$$  

This gives the same equation as before.

Note that the fixed point of $T$ lies on both circles.
The square and its transform, a square of double the area
The red circle transforms to the cyan circle
The point $(4, 3)$ where the circles meet in the first quadrant is the fixed point
Question 6

Consider the function \( f(z) = (z - 4)^{\frac{1}{2}} \), where \( f(8) = 2 \).

If the branch cut for \( f \) is the interval on the real axis \((-\infty, 4]\), determine the value of the following, explaining your results:

- \( f(4i) \)
- \( \lim_{t \to 0^+} f(ti) \)
- \( \lim_{t \to 0^-} f(t(1 + i)) \)

If instead the branch cut is along the line \( z = 4 + si \), for \( s \) real and non-negative, which, if any, of these values changes?

Explain your answer.

We write \( z - 4 = se^{i\alpha} \), where \( s > 0 \) is the distance of the point \( z \) from the point 4 and \( \alpha \) radians is the angle that the vector from 4 to \( z \) makes with the positive \( x \)-axis.

Then \( f(z) = (z - 4)^{\frac{1}{2}} = \sqrt{s} e^{i\frac{\alpha}{2}} \).
With the cut along the real axis on the interval \((-\infty, 4]\), we may take \(\alpha\) in the range \(-\pi < \alpha < \pi\), since then we have for the point \(z = 8, s = |z - 4| = 8 - 4| = 4\) and \(\alpha = 0\) and then \(f(8) = \sqrt{4}e^0 = 2\), as required.

Then we have:

- For \(f(4i)\), we have \(z - 4 = 4i - 4\), so \(s = \sqrt{16 + 16} = 2\sqrt{2}\).
  
  Also \(\alpha = \frac{3\pi}{4}\), so we have:

\[
f(4i) = 2\sqrt[4]{2} e^{\frac{3i\pi}{8}} = 2\sqrt[4]{2} \left( \cos \left( \frac{3i\pi}{8} \right) + i \sin \left( \frac{3i\pi}{8} \right) \right) = 0.91018 + 2.19737i.
\]

Alternatively, we algebraically solve the equation:

\[
z^2 = 4i - 4 = x^2 - y^2 + 2ixy, \quad z = x + iy, \quad x^2 - y^2 = -4, \quad xy = 2, \quad |z|^2 = |z^2| = |4i - 4| = \sqrt{32} = x^2 + y^2,
\]

\[
x^2 = \frac{1}{2}(\sqrt{32} - 4) = 2(\sqrt{2} - 1), \quad y^2 = \frac{1}{2}(\sqrt{32} + 4) = 2(\sqrt{2} + 1).
\]

The desired solution lies in the first quadrant, since the angle \(\frac{\alpha}{2} = \frac{3\pi}{8}\) radians is an acute angle. So we get:

\[
f(4i) = \sqrt{2} \left( \sqrt{\sqrt{2} - 1} + i\sqrt{\sqrt{2} + 1} \right) = 0.91018 + 2.19737i.
\]

- \(\lim_{t \to 0^+} f(t)i\)

Here we see that \(s = \lim_{t \to 0^+} |ti - 4| = 4\).

Also since we are approaching the origin from above, we see that the argument of \(z - 4\) goes to \(\pi\), so in the limit, we have \(\alpha = \pi\).

So we get:

\[
\lim_{t \to 0^+} f(t)i = \sqrt{4}e^{\frac{i\pi}{2}} = 2i.
\]

- \(\lim_{t \to 0^-} f(t(1 + i))\)

Here we see that \(s = \lim_{t \to 0^-} |t(1 + i) - 4| = 4\).

Also since we are approaching the origin from below, we see that the argument of \(z - 4\) goes to \(-\pi\), so in the limit, we have \(\alpha = -\pi\).

So we get:

\[
\lim_{t \to 0^-} f(t(1 + i)) = \sqrt{4}e^{-\frac{i\pi}{2}} = -2i.
\]

Note that in this case the function cannot be continuous across the cut.
With the cut along the line $z = 4 + si$, for $s$ real and non-negative, we may take $\alpha$ in the range $-\frac{3\pi}{2} < \alpha < \frac{\pi}{2}$, since then we have for the point $z = 8$, $s = |z - 4| = |8 - 4| = 4$ and $\alpha = 0$ and then $f(8) = \sqrt{4} e^{0} = 2$, as required. Then we have:

- For $f(4i)$, we have $z - 4 = 4i - 4$, so $s = \sqrt{16 + 16} = 2\sqrt{2}$.
  Also $\alpha = -\frac{5\pi}{4}$, so we have:
  
  $$f(4i) = 2\sqrt{2} e^{\frac{5\pi}{8}}.$$  

  This lies in the third quadrant and is the negative of our previous solution, so we now have:

  $$f(4i) = -\sqrt{2} \left( \sqrt{2} - 1 + i \sqrt{2} + 1 \right) = -0.91018 - 2.19737i.$$  

- $\lim_{t \to 0^+} f(t(1 + i))$
  Here we see that $s = \lim_{t \to 0^+} |t(1 + i) - 4| = 4$.
  Also since we are approaching the origin from below, we see that the argument of $z - 4$ goes to $-\pi$, so in the limit, we have $\alpha = -\pi$.
  So we get:
  
  $$\lim_{t \to 0^+} f(t(1 + i)) = \sqrt{4} e^{\frac{-i\pi}{2}} = -2i.$$  

- $\lim_{t \to 0^-} f(t(1 + i))$
  Here we see that $s = \lim_{t \to 0^-} |t(1 + i) - 4| = 4$.
  Also since we are approaching the origin from below, we see that the argument of $z - 4$ goes to $-\pi$, so in the limit, we have $\alpha = -\pi$.
  So we get:
  
  $$\lim_{t \to 0^-} f(t(1 + i)) = \sqrt{4} e^{\frac{-i\pi}{2}} = -2i.$$  

Note that in this case $f$ is analytic near the origin.
Question 7

Write the function \( g = \frac{4x - 8y}{x^2 + y^2} \) as the real part of an analytic function.

Hint: consider the function \( z^{-1} \).

Hence, or otherwise, solve the equation:

\[
\left( \partial_x^2 + \partial_y^2 \right) f = g.
\]

We have, with \( z = x + iy \):

\[
\frac{1}{z} = \frac{x - iy}{x^2 + y^2}, \quad i \frac{1}{z} = \frac{y + ix}{x^2 + y^2},
\]

\[
\frac{4 - 8i}{z} = \frac{4x - 8y + i(-4y - 8x)}{x^2 + y^2},
\]

So if we write: \( G(z) = (4 - 8i)z^{-1} \), then \( G \) is analytic and \( g = \Re(G) \), as required.

To solve the equation: \( \left( \partial_x^2 + \partial_y^2 \right) f = g \), we first solve the equation \( \left( \partial_x^2 + \partial_y^2 \right) F = G \), and then \( f = \Re(F) \).

So we solve:

\[
4\partial_z \partial_{\overline{z}} F = \frac{4 - 8i}{z},
\]

\[
\partial_z \partial_{\overline{z}} F = \frac{1 - 2i}{z},
\]

Integrating both sides of this equation with respect to \( z \) (ignoring integration constants), treating \( z \) as a constant, gives:

\[
\partial_z F = (1 - 2i) \ln(z).
\]

Integrating both sides of this equation with respect to \( \overline{z} \) (again ignoring integration constants), treating \( z \) as a constant, gives the solution:

\[
F = (1 - 2i) \overline{z} \ln(z).
\]

The general local solution is then \( F = (1 - 2i) \overline{z} \ln(z) + H(z) + J(\overline{z}) \), where \( H \) is analytic and \( J \) is anti-analytic.

So the required solution is:

\[
f = \Re((1 - 2i) \overline{z} \ln(z) + m(z)).
\]

Here \( m(z) \) is an arbitrary analytic function.
To write out the function $f$ explicitly, in terms of $x$ and $y$, we need:

$$\Re((1 - 2i)z \ln(z)) = \Re((1 - 2i)(x - iy)(\ln(r) + i\theta))$$

$$= \Re((x - 2y - i(2x + y))(\ln(r) + i\theta))$$

$$= (x - 2y) \ln(r) + (2x + y)\theta$$

$$= \frac{1}{2}(x - 2y) \ln(x^2 + y^2) + (2x + y)\arctan\left(\frac{y}{x}\right).$$

Here we have written $z = re^{i\theta}$ with $r = \sqrt{x^2 + y^2} > 0$ and $\theta = \arctan\left(\frac{y}{x}\right)$.

Also we have used the fact that $\ln(r) = \frac{1}{2} \ln(r^2) = \frac{1}{2} \ln(x^2 + y^2)$.

So the required local solution is:

$$f = \frac{1}{2}(x - 2y) \ln(x^2 + y^2) + (2x + y)\arctan\left(\frac{y}{x}\right) + M(x,y).$$

Here $M$ is an arbitrary harmonic function, or, equivalently, the real part of an analytic function.

A somewhat simpler solution is to replace the $(1 - 2i)z \ln(z)$ term of the solution by the term $(1 - 2i)\frac{z}{\bar{z}} \ln\left(\frac{\bar{z}}{z}\right)$, which is legitimate, because the extra term, which is $-(1 - 2i)z \ln(z)$, is anti-analytic so is harmonic.

Then, compared to the above calculation, the $\theta$ term doubles and the $r$ term goes away, so now we have:

$$f = \Re\left((1 - 2i)\frac{z}{\bar{z}} \ln\left(\frac{\bar{z}}{z}\right) + n(z)\right) = 2(2x + y)\arctan\left(\frac{y}{x}\right) + N(x,y).$$

Here $n$ is analytic and $N$ is harmonic.