

Motion in a Newtonian potential in three dimensions

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The conservation of energy

Let \underline{r} be the position vector of the particle at time t .

Put $\underline{v} = \frac{d\underline{r}}{dt}$ and $\underline{a} = \frac{d\underline{v}}{dt}$.

So \underline{v} and \underline{a} are the velocity and acceleration of the particle, respectively.

Let $r = |\underline{r}|$, so r is the distance of the particle from the origin.

We assume that $r > 0$, so $\underline{r} \neq \underline{0}$ and the particle is away from the origin.

Let $\underline{R} = r^{-1}\underline{r}$, so $\underline{R} \cdot \underline{R} = 1$ and \underline{R} is the unit vector pointing to the particle.

Put $v = |\underline{v}|$, so v is the speed of the particle.

Then motion in a central potential is given by the equation:

$$\underline{a} = f'(r)\underline{R} = r^{-1}f'(r)\underline{r}.$$

Here f is the potential function, and is only defined up to a constant.

If the force goes to zero at infinity, it is usual to fix the constant by requiring that f go to zero at infinity also.

The Newtonian case is $f = kr^{-1}$, so $f'(r) = -kr^{-2}$, the inverse square law.

Here k is a constant which we will always take to be non-zero.

The force is attractive if $k > 0$ and repulsive if $k < 0$.

Then we have:

$$\frac{d}{dt}(v^2) = \frac{d(\underline{v} \cdot \underline{v})}{dt} = 2\underline{v} \cdot \underline{a} = -2kr^{-2}\underline{R} \cdot \underline{v}.$$

Also we have the relation:

$$r \frac{dr}{dt} = \frac{1}{2} \frac{d(r^2)}{dt} = \frac{1}{2} \frac{d(\underline{r} \cdot \underline{r})}{dt} = \underline{r} \cdot \underline{v},$$

$$\frac{dr}{dt} = r^{-1} \underline{r} \cdot \underline{v} = \underline{R} \cdot \underline{v},$$

$$\frac{d}{dt}(r^{-1}) = -r^{-2} \frac{dr}{dt} = -r^{-2} \underline{R} \cdot \underline{v}.$$

So we may define E by the relation:

$$E = v^2 - 2kr^{-1}.$$

Then we have that E is conserved:

$$\frac{dE}{dt} = \frac{d}{dt}(v^2) - 2k \frac{d}{dt}(r^{-1}) = 0.$$

Also we have the inequality $E + 2kr^{-1} = v^2 \geq 0$.

Finally note that if E is negative, then k is necessarily positive.

Angular momentum conservation; Kepler's Second Law

The angular momentum vector per unit mass, \underline{J} is:

$$\underline{J} = \underline{r} \times \underline{v}.$$

We have:

$$\frac{d\underline{J}}{dt} = \frac{d\underline{r}}{dt} \times \underline{v} + \underline{r} \times \frac{d\underline{v}}{dt} = \underline{v} \times \underline{v} + f'(r)r^{-1}\underline{r} \times \underline{r} = 0.$$

So \underline{J} is conserved. We put $J = |\underline{J}| = \sqrt{\underline{J} \cdot \underline{J}} = \sqrt{r^2 v^2 - (\underline{r} \cdot \underline{v})^2}$.
So J is constant and is non-negative.

If $J = 0$, then $\underline{v} \times \underline{r} = 0$, so since $\underline{r} \neq \underline{0}$, we get $\underline{v} = u(t)\underline{r}$, for some function $u(t)$. Let $m(t) > 0$ obey the equation:

$$\frac{dm(t)}{dt} = -u(t)m(t), \text{ so } m(t) = e^{-\int u(t)dt}.$$

Then we have: $\frac{d}{dt}(m(t)\underline{r}) = \left(\frac{dm}{dt}\right)\underline{r} + m(t)\underline{v} = \left(\frac{dm}{dt} + m(t)u(t)\right)\underline{r} = \underline{0}$.

So $m(t)\underline{r} = \underline{r}_0$, a constant non-zero vector. So \underline{r} points always in a fixed direction and we may write $\underline{r} = r(t)\underline{n}$, where $r(t) > 0$ and \underline{n} is a constant unit vector. In particular the motion is radial, when $J = 0$.

If $d\underline{r}$ is an infinitesimal displacement, then the displacement traces out an area dA , when joined to the force center, given by the formula:

$$2dA = |\underline{r} \times d\underline{r}| = |\underline{r} \times \underline{v}|dt = Jdt.$$

So the area $A(t)$, traced out by the radius vector, increases linearly with time:

$$2A(t) = J(t - t_0).$$

Here t_0 is an initial reference time.

This is Kepler's Second law.

In particular if the trajectory is a closed orbit of area A , and the orbital period is T , we have the relation:

$$2A = JT.$$

When $J = 0$ the motion is linear, so, a fortiori, is also planar.

When $J \neq 0$, the motion lies in the plane through the origin with equation $\underline{J} \cdot \underline{r} = 0$.

So in all cases, the motion is planar.

The radial motion as a function of time

We have $\underline{J} = \underline{r} \times \underline{v}$, which gives the relation:

$$\begin{aligned} J^2 &= |\underline{J}|^2 = |\underline{r} \times \underline{v}|^2 = (\underline{r} \cdot \underline{r})(\underline{v} \cdot \underline{v}) - (\underline{r} \cdot \underline{v})^2 \\ &= r^2(E + 2kr^{-1}) - (\underline{r} \cdot \underline{v})^2 = Er^2 + 2kr - (\underline{r} \cdot \underline{v})^2. \\ r^2 \left(\frac{dr}{dt} \right)^2 &= (\underline{r} \cdot \underline{v})^2 = r^2E + 2kr - J^2. \end{aligned}$$

The discriminant of the quadratic $r^2E + 2kr - J^2$ is $4(k^2 + J^2E)$. Since $(\underline{r} \cdot \underline{v})^2 \geq 0$, if $E < 0$, the quadratic must have at least one real root, so we have the inequality, valid in all cases:

$$k^2 + J^2E \geq 0.$$

The equation $r^2 \left(\frac{dr}{dt} \right)^2 = r^2E + 2kr - J^2$ can be solved to give a relation between r and t . Here $r > 0$, $k \neq 0$, $J \geq 0$.

If desired, we can write the equation in terms of the speed, v , since we have $v^2 - 2kr^{-1} = E$, so $r = \frac{2k}{v^2 - E}$ (so in particular if $k > 0$ then $v^2 > E$ and if $k < 0$, then $v^2 < E$).

Then the evolution equation becomes:

$$\begin{aligned} 64k^4v^2 \left(\frac{dv}{dt} \right)^2 &= (v^2 - E)^6 (r^2E + 2kr - J^2) \\ &= (v^2 - E)^4 (4k^2E + 4k^2(v^2 - E) - J^2(v^2 - E)^2), \\ \epsilon 8k^2v \frac{dv}{dt} &= (v^2 - E)^2 \sqrt{-J^2v^4 + (4k^2 + 2EJ^2)v^2 - J^2E^2}. \end{aligned}$$

Here and in the following, $\epsilon = \pm 1$ and is such that $\epsilon \frac{dv}{dt}$ is non-negative.

The discriminant of the quadratic in v^2 inside the square root on the right-hand-side of the equation is $(4k^2 + 2EJ^2)^2 - 4J^4E^2 = 16k^2(k^2 + EJ^2)^2$, which is non-negative.

Note that $\frac{dr}{dt} = 4kv^2(v^2 - E)^{-2} \frac{dv}{dt}$, so the quantities $\frac{dr}{dt}$ and $k \frac{dv}{dt}$ have the same sign.

We treat the evolution equation case by case, first considering the cases that $J = 0$, for which the motion is purely radial and is classified by the total energy E :

- If $E = J = 0$, then $k > 0$ and we have:

$$64k^4v^2 \left(\frac{dv}{dt} \right)^2 = v^8(4k^2v^2),$$

$$4kv' = \epsilon v^4, \quad -4kv^{-4}dv = \epsilon dt,$$

$$3\epsilon(t - t_0) = 4kv^{-3}, \quad r = 2kv^{-2},$$

$$r = \left(\frac{9k}{2} \right)^{\frac{1}{3}} (t - t_0)^{\frac{2}{3}}, \quad v = \epsilon \left(\frac{4k}{3} \right)^{\frac{1}{3}} (t - t_0)^{-\frac{1}{3}}.$$

Note that $v \rightarrow 0$ as $|t| \rightarrow \infty$.

So when $E = J = 0$, the motion (which is radial) either starts at infinity in the infinite past, and ends as $t \rightarrow t_0^-$, when it crashes into the singularity at $r = 0$ at infinite speed in finite time, always speeding up, or the motion emerges from the origin at infinite speed as $t \rightarrow t_0^+$ and goes out to infinity always slowing down, with limiting speed zero as $t \rightarrow \infty$.

- If $J = 0$ and $E > 0$, then we have:

$$\epsilon 8k^2 v \frac{dv}{dt} = (v^2 - E)^2 \sqrt{4k^2 v^2} = 2|k|v(v^2 - E)^2,$$

$$\epsilon(t-t_0) = - \int \frac{4|k|dv}{(v^2 - E)^2} = -4|k| \frac{\partial}{\partial E} \int \frac{dv}{v^2 - E} = \frac{2v|k|}{E(v^2 - E)} - \frac{|k|}{E^{\frac{3}{2}}} \ln \left| \frac{v + \sqrt{E}}{v - \sqrt{E}} \right|,$$

$$r = \frac{2k}{v^2 - E}.$$

Write:

$$v = w\sqrt{E}, \quad r = 2|k|E^{-1}s, \quad t = |k|^{-1}E^{\frac{3}{2}}, \quad t_0 = |k|^{-1}E^{\frac{3}{2}}u_0, \quad \zeta = k|k|^{-1}.$$

Note that s is positive, $w \geq 0$ and $\zeta = \pm 1$ is the sign of k .

Then we have the solution in universal parametric form:

$$\zeta s = \frac{2}{w^2 - 1}, \quad \epsilon(u - u_0) = \frac{2w}{w^2 - 1} - \ln \left| \frac{w + 1}{w - 1} \right|.$$

- If $k > 0$, so $\zeta = 1$, then $w^2 > 1$, so $w > 1$, or $w < -1$.
 - * If $w > 1$, the solution goes to infinity as $w \rightarrow 1^+$, when $\epsilon u \rightarrow \infty$.
The speed is bounded away from zero.
Then as $w \rightarrow \infty$, the solution goes to the origin at infinite speed at finite time.
 - * If $w < -1$, the solution goes to infinity as $w \rightarrow -1^-$, when $\epsilon t \rightarrow -\infty$.
The speed is bounded away from zero.
Then as $w \rightarrow -\infty$, the solution goes to the origin at infinite speed, at finite time.
- If $k < 0$, then $w^2 < 1$, so $-1 < w < 1$.
Then the solution goes to infinity as $w \rightarrow -1^+$ and as $w \rightarrow 1^-$, always with bounded speed and in the infinite future and past.
The solution comes in from infinity in the infinite past reaches a minimum positive r value of $-2kE^{-1}$ and then turns around and goes out to infinity again in the infinite future.
The speed is bounded above by \sqrt{E} , which it reaches asymptotically.

- If $J = 0$ and $E < 0$, put $E = -F^2$, where $F > 0$.
Then we have:

$$\begin{aligned}\epsilon(t-t_0) &= - \int \frac{4kdv}{(v^2 + F^2)^2} = \frac{2k}{F} \frac{\partial}{\partial F} \int \frac{dv}{v^2 + F^2} = \frac{2k}{F} \frac{\partial}{\partial F} \frac{1}{F} \arctan(vF^{-1}) \\ &= -\frac{2k}{F^3} \arctan(vF^{-1}) - \frac{2kv}{F^2(v^2 + F^2)}, \\ r &= \frac{2k}{v^2 + F^2}.\end{aligned}$$

Note that k is necessarily positive.

Write:

$$v = w(-E)^{\frac{1}{2}}, \quad r = -2kE^{-1}s, \quad t = (-E)^{\frac{3}{2}}k^{-1}u, \quad t_0 = (-E)^{\frac{3}{2}}k^{-1}u_0.$$

Then we have the solution in universal form:

$$\begin{aligned}s &= \frac{1}{1 + w^2}, \\ \epsilon(u - u_0) &= -2 \arctan(w) - \frac{2w}{1 + w^2}.\end{aligned}$$

The solution leaves the origin with infinite speed in the finite past, slows down rapidly to reach a maximum radius of $-2kE^{-1}$ and then goes back to the origin with infinite speed in the finite future.

The solution lasts for a total time of $2k^{-1}(-E)^{\frac{3}{2}}\pi$.

Next we consider the last case where $E = 0$:

- When $E = 0, J > 0$, then $k > 0$ and the evolution equation becomes:

$$r^2 \left(\frac{dr}{dt} \right)^2 = 2kr - J^2,$$

$$\begin{aligned} \epsilon 3k^2(t - t_0) &= \int \frac{3k^2 r dr}{\sqrt{2kr - J^2}} = 3kr\sqrt{2kr - J^2} - \int 3k\sqrt{2kr - J^2} dr \\ &= (3kr - (2kr - J^2))\sqrt{2kr - J^2} = (kr + J^2)\sqrt{2kr - J^2}, \\ 9k(t - t_0)^2 &= 2(r - r_0)(r + 2r_0)^2, \quad r_0 = \frac{J^2}{2k} > 0. \end{aligned}$$

This is a cubic in r , which can be solved to give r explicitly in terms of t .

Note that this means there is a complex torus underlying the evolution. Since the left-hand side is non-negative, the radius is bounded below by r_0 which it achieves at time t_0 .

As $t \rightarrow \pm\infty, r \rightarrow \infty$.

By rescaling the variables, we have a universal cubic curve:

$$\begin{aligned} 4y^2 &= (x - 1)(x + 2)^2, \\ x &= rr_0^{-1} = 2krJ^{-2}, \\ y &= \epsilon 3k^{\frac{1}{2}}(2r_0)^{-\frac{3}{2}}(t - t_0) = 3k^2 J^{-3} \epsilon(t - t_0). \end{aligned}$$

Using Maple, the solution of the cubic for x in terms of y reads:

$$x = \left(1 + 2y^2 + 2\sqrt{y^2 + y^4} \right)^{\frac{1}{3}} + \left(1 + 2y^2 + 2\sqrt{y^2 + y^4} \right)^{-\frac{1}{3}} - 1.$$

In these variables, the original differential equation becomes just:

$$16(x - 1) = 9x^2(dx/dy)^2.$$

It is easily verified that the solution x does obey this equation.

The plot shows that x comes in from infinity, for large negative y , goes to its minimum value of $x = 1$ at $y = 0$ and then goes out to infinity again as y goes to infinity.

We are left with the two generic cases: $J > 0, E < 0$ and $J > 0, E > 0$.

- First we take $J > 0$ and $E < 0$.

The equation to be analyzed is:

$$r^2 \left(\frac{dr}{dt} \right)^2 = r^2 E + 2kr - J^2.$$

Necessarily we have $k > 0$, since otherwise the right-hand side is negative.

Solving the quadratic, we have the factorization:

$$r^2 \left(\frac{dr}{dt} \right)^2 = E(r - r_-)(r - r_+),$$

$$r_{\pm} = \frac{k \pm \sqrt{k^2 + J^2 E}}{-E}.$$

Note that since $E < 0$, we have $0 < r_- \leq r_+$ and r is confined to the closed interval $[r_-, r_+]$.

- In the particular case that $k^2 + J^2 E = 0$, the quadratic is a perfect square, so $r_+ = r_- = k(-E)^{-1} = J^2 k^{-1}$, so the motion is circular and uniform, so is simple harmonic motion on the circle, with period given by Kepler's formula:

$$k^2 T = 2\pi J^3.$$

$$kT^2 = 4\pi^2 J^6 k^{-3} = 4\pi a^2.$$

Here $a = r_+ = r_-$.

So we may henceforth assume that $k^2 + JE > 0$, so $r_- < r_+$.

Put:

$$a = \frac{1}{2}(r_+ + r_-) = -kE^{-1},$$

$$b = \sqrt{r_+ r_-} = -JE^{-\frac{1}{2}}.$$

$$c = \sqrt{a^2 - b^2} = \frac{1}{2}(r_+ - r_-) = -E^{-1} \sqrt{k^2 + J^2 E} > 0.$$

Then we have:

$$r^2 \left(\frac{dr}{dt} \right)^2 = E(r - r_-)(r - r_+) = E(r^2 - (r_+ + r_-)r + r_+ r_-)$$

$$= E(r^2 - 2ar + b^2) = E((r - a)^2 + b^2 - a^2) = E((r - a)^2 - c^2).$$

Now put $r = cs + a$, $t(-E)^{\frac{1}{2}} = au$ and $t_0(-E)^{\frac{1}{2}} = au_0$, where t_0 is an initial time.

Note that when $s = 1$, $r = a + c = r_+$ and when $s = -1$, $r = a - c = r_-$. Finally put $c = \epsilon a$, so $0 < \epsilon = k^{-1}\sqrt{EJ^2 + k^2} = \sqrt{1 + EJ^2k^{-2}} < 1$. Then we have $|s| \leq 1$ and:

$$-E(1 - s^2) = a^2(1 + \epsilon s)^2 \left(\frac{ds}{dt} \right)^2,$$

$$1 - s^2 = (1 + \epsilon s)^2 \left(\frac{ds}{du} \right)^2,$$

$$\zeta(u - u_0) = \int \frac{(1 + \epsilon s)ds}{\sqrt{1 - s^2}} = -\epsilon\sqrt{1 - s^2} + \arcsin(s).$$

Here $\zeta^2 = 1$.

Put $s = \sin(w)$, with $|w| \leq \frac{\pi}{2}$.

Then we have:

$$\zeta(u - u_0) = -\epsilon \cos(w) + w.$$

This equation was discovered by Johannes Kepler.

Note that as s goes from -1 to 1 , w goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

The function $-\epsilon \cos(w) + w$ is monotonically strictly increasing with w and goes from the values $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ also.

So as time passes (taking $\zeta = 1$, for convenience) starting from $u = u_0 - \frac{\pi}{2}$ at which point $r = r_-$, r increases steadily ending at $u = u_0 + \frac{\pi}{2}$ at which point $r = r_+$.

Then the motion turns around and r decreases back to r_- after u increases by an additional π .

Then the motion repeats.

So it is periodic with u period 2π , so the time is:

$$T = 2\pi a(-E)^{-\frac{1}{2}} = 4\pi k(-E)^{-\frac{3}{2}} = 4\pi k^{-\frac{1}{2}}a^{\frac{3}{2}}.$$

Then we have Kepler's Third Law:

$$kT^2 = 4\pi^2 a^3.$$

Note that measuring T and a and using this law gives a direct determination of the strength of the force law.

Our final generic case is:

- We now take $J > 0, E > 0$.
The equation to be analyzed is:

$$r^2 \left(\frac{dr}{dt} \right)^2 = r^2 E + 2kr - J^2.$$

Here k can be positive or negative.

Solving the quadratic, we have the factorization:

$$r^2 \left(\frac{dr}{dt} \right)^2 = E(r + r_-)(r - r_+),$$

$$r_{\pm} = \frac{\sqrt{k^2 + J^2 E} \pm k}{E}.$$

Note that we have r_+ and r_- both positive and the motion therefore has $r \geq r_+$ always.

Put:

$$a = \frac{1}{2}(r_+ - r_-) = kE^{-1},$$

$$b = \sqrt{r_+ r_-} = JE^{-\frac{1}{2}}.$$

$$c = \sqrt{a^2 + b^2} = \frac{1}{2}(r_+ + r_-) = E^{-1} \sqrt{k^2 + J^2 E} > 0.$$

Note that $c > |a| > 0$.

Then we have:

$$\begin{aligned} r^2 \left(\frac{dr}{dt} \right)^2 &= E(r + r_-)(r - r_+) = E(r^2 - (r_+ - r_-)r - r_+ r_-) \\ &= E(r^2 - 2ar - b^2) = E((r - a)^2 - b^2 - a^2) = E((r - a)^2 - c^2). \end{aligned}$$

Now put $r = cs + a = c(s - 1) + r_+$, $E^{\frac{1}{2}}t = au$ and $t_0 E^{\frac{1}{2}}t_0 = au_0$, where t_0 is an initial time.

Note that $s \geq 1$ and when $s = 1$, $r = r_+$.

Finally put $c = \epsilon a$, so $|\epsilon| = k^{-1} \sqrt{EJ^2 + k^2} = \sqrt{1 + EJ^2 k^{-2}} > 1$.

Then the differential equation becomes:

$$(cs + a)^2 \left(\frac{ds}{dt} \right)^2 = E(s^2 - 1),$$

$$\frac{(\epsilon s + 1)ds}{\sqrt{s^2 - 1}} = \zeta du,$$

$$\zeta(u - u_0) = \epsilon \sqrt{s^2 - 1} + \ln(s + \sqrt{s^2 - 1}).$$

Here $\zeta = \pm 1$.

Note that for $s > 1$, the term $\sqrt{s^2 - 1}$ is always larger than the term $\ln(s + \sqrt{s^2 - 1})$, so the right-hand-side has the sign of k .

Note that if $u > u_0$, ζ and k have the same sign, whereas if $u < u_0$, ζ and k have opposite signs.

When $s = 1$, we have $u = u_0$, so our initial time is when the radius is at its minimum value r_+ .

Then the time is monotonic in r for $r > r_+$.

The motion starts at infinite radius in the infinite past, comes into the minimal radius r_+ at $t = t_0$ and then goes out to infinity again in the infinite future.

The trajectories when the angular momentum is non-zero

We consider the trajectories in the case that the motion has non-zero angular momentum, so $J > 0$.

Then $\underline{J} \neq 0$ and the motion lies entirely in the plane $\underline{r} \cdot \underline{J} = 0$.

Also, since $J^2 = r^2 v^2 - (\underline{r} \cdot \underline{v})^2 > 0$, this forces r and v to be strictly positive, so, in particular, the motion is never at rest and $E + 2kr^{-1} = v^2 > 0$.

The same formula, rewritten, gives the relation:

$$|\underline{r} \cdot \underline{v}| = \sqrt{r^2 v^2 - J^2} = \sqrt{r^2 E + 2rk - J^2}.$$

In particular the motion always has the property that $r^2 E + 2rk - J^2 \geq 0$. Note that regarded as a quadratic in r , this has discriminant $4(k^2 + EJ^2)$.

Next define the vector \underline{s} by the formula, valid since J and r are positive:

$$\underline{s} = (\underline{v} \cdot \underline{v}) \underline{r} - 2(\underline{r} \cdot \underline{v}) \underline{v} = v^2 \left(\underline{r} - \frac{2(\underline{r} \cdot \underline{v}) \underline{v}}{\underline{v} \cdot \underline{v}} \right).$$

Geometrically \underline{s} is v^2 times the reflection of \underline{r} in the normal line to the motion, which is perpendicular to \underline{v} . So we have $|\underline{s}| = v^2 r$. Also we have:

$$\begin{aligned} \frac{d\underline{s}}{dt} &= \frac{d}{dt} (\underline{v} \cdot \underline{v} \underline{r} - 2(\underline{r} \cdot \underline{v}) \underline{v}) = 2(\underline{v} \cdot \underline{a}) \underline{r} + v^2 \underline{v} - 2v^2 \underline{v} - 2(\underline{r} \cdot \underline{a}) \underline{v} - 2(\underline{r} \cdot \underline{v}) \underline{a} \\ &= 2kr^{-3} (\underline{v} \cdot \underline{r}) \underline{r} - v^2 \underline{v} + 2kr^{-3} (\underline{r} \cdot \underline{r}) \underline{v} + 2kr^{-3} (\underline{r} \cdot \underline{v}) \underline{r} \\ &= -(v^2 - 2kr^{-1}) \underline{v} = -E \underline{v} = -\frac{d}{dt} (E \underline{r}). \end{aligned}$$

Accordingly we define:

$$\underline{H} = \underline{s} + E \underline{r}.$$

Note that $\underline{H} \cdot \underline{J} = 0$, so \underline{H} lies in the plane of motion.

Then we have $\frac{d\underline{H}}{dt} = \underline{0}$, so \underline{H} is conserved. Note also that we have:

$$|\underline{s}| = v^2 r = (E + 2kr^{-1}) r, \quad |\underline{s}| - E|\underline{r}| = 2k.$$

The two equations $\underline{H} = \underline{s} + E\underline{r}$ and $|\underline{s}| - E|\underline{r}| = 2k$ show that the trajectory is a conic:

- If $E > 0$ then the trajectory is an hyperbola with foci at the origin and at $\underline{h} = E^{-1}\underline{H}$.
- If $E < 0$, then the trajectory is an ellipse with foci at the origin and at $\underline{h} = E^{-1}\underline{H}$.
- If $E = 0$, then the trajectory is a parabola, with one focus at the origin and axis parallel to \underline{H} .

Note that at the vertex of the parabola, say $\underline{r} = \underline{r}_0$, the motion is perpendicular to the radius vector, so $\underline{r}_0 \cdot \underline{v} = 0$, so $\underline{s} = v^2 \underline{r}_0 = 2k|\underline{r}_0|^{-1} \underline{r}_0$, so points along the axis, as expected.

This is not a geometrical vector, since it has dimensions of energy times length, so its length does not represent a physical length such as the length to the directrix.

Instead from work below, the distance to the directrix is $2|\underline{r}_0| = k^{-1}J^2$.

Next we have:

$$\begin{aligned} \underline{s} \cdot \underline{r} &= \underline{r} \cdot ((\underline{v} \cdot \underline{v})\underline{r} - 2(\underline{r} \cdot \underline{v})\underline{v}) \\ &= v^2 r^2 - 2(\underline{r} \cdot \underline{v})^2 = 2J^2 - v^2 r^2 = 2J^2 - (E + 2kr^{-1})r^2 = 2J^2 - Er^2 - 2kr, \\ \underline{H} \cdot \underline{r} &= (\underline{s} + E\underline{r}) \cdot \underline{r} = 2J^2 - Er^2 - 2kr + Er^2 = 2J^2 - 2kr. \end{aligned}$$

Finally we have:

$$\begin{aligned} \underline{H} \cdot \underline{H} &= (\underline{s} + E\underline{r}) \cdot (\underline{s} + E\underline{r}) \\ &= \underline{s} \cdot \underline{s} + 2E\underline{s} \cdot \underline{r} + E^2 \underline{r} \cdot \underline{r} \\ &= v^4 r^2 + 2E(2J^2 - Er^2 - 2kr) + E^2 r^2 \\ &= r^2(E + 2kr^{-1})^2 + 2E(2J^2 - Er^2 - 2kr) + E^2 r^2 \\ &= (Er + 2k)^2 + 4EJ^2 - E^2 r^2 - 4Ekr \\ &= 4(k^2 + EJ^2). \end{aligned}$$

In particular we have the inequality $k^2 + EJ^2 \geq 0$.

Let the angle between \underline{H} and \underline{r} be θ .

Then we have:

$$\begin{aligned} 2J^2 - 2kr &= \underline{H} \cdot \underline{r} = |\underline{H}| |\underline{r}| \cos(\theta), \\ r(k + \sqrt{EJ^2 + k^2} \cos(\theta)) &= J^2, \\ r(1 + \epsilon \cos(\theta)) &= \lambda, \\ \epsilon &= k^{-1} \sqrt{EJ^2 + k^2}, \quad \lambda = k^{-1} J^2. \end{aligned}$$

This is the standard polar equation of a conic section of eccentricity $|\epsilon|$ and with one focus at the origin and with the axis of symmetry in the direction of \underline{H} .

Note that we have:

$$\epsilon^2 = 1 + \frac{EJ^2}{k^2}.$$

The conic is:

- a parabola, if and only if $|\epsilon| = 1$, if and only if $E = 0$;
- an hyperbola if and only if $|\epsilon| > 1$, if and only if $E > 0$;
- an ellipse or circle if and only if $|\epsilon| < 1$, if and only if $E < 0$.

The conic is a circle only in the case that $E = -\frac{k^2}{J^2}$, which is the smallest possible energy for a given k and given angular momentum.

Summarizing, we have proved a refined version of Kepler's First Law:

- For any value of the angular momentum, the trajectories are bounded if and only if the total energy is negative.
- The trajectories are radial if and only if the angular momentum is zero.
- The trajectories with non-zero angular momentum form conics, with one focus at the force center.
- The conic is an ellipse or circle if and only if the total energy is negative, an hyperbola, if and only if the total energy is positive and a parabola if and only if the total energy is zero.

In the case that $E < 0$, we have:

$$r_- = \frac{\lambda}{1 + |\epsilon|}, \quad r_+ = \frac{\lambda}{1 - |\epsilon|}.$$

Then the semi-major axis a is:

$$a = \frac{1}{2}(r_+ + r_-) = \frac{\lambda}{1 - \epsilon^2} = k(-E)^{-1}.$$

The semi-minor axis is given by:

$$b = \sqrt{r_+ r_-} = \frac{\lambda}{\sqrt{1 - \epsilon^2}} = J(-E)^{-\frac{1}{2}}.$$

The distance d to the other focus is:

$$d = \frac{1}{r_+ - r_-} = \frac{\lambda|\epsilon|}{1 - \epsilon^2} = \frac{\sqrt{EJ^2 + k^2}}{-E}.$$

This agrees with the length of $\underline{h} = E^{-1}\underline{H}$ as expected.

The area of the ellipse is given by:

$$A = \pi ab = \pi k J(-E)^{-\frac{3}{2}}.$$

In particular we the relation, giving the orbital period:

$$T = \frac{2A}{J} = 2\pi k(-E)^{-\frac{3}{2}},$$

$$kT^2 = 4\pi^2 k^3 (-E)^{-3} = 4\pi^2 a^3.$$

This is Kepler's Third Law.