Putnam Seminar:
First problem set with hints 9/13/8

Question 1

Find with proof the range of the following functions:

- $\cos(x) - \cos(x+1)$

  Hint: trigonometric identities.

- $\cos(x^2) - \cos((x+1)^2)$

  Hint: there is no hope here unless the answer is very simple!

- $\dfrac{\sin(x)}{x}$

Question 2

How many positive integers n are there, such that n is an exact divisor of at least one of the integers 10^{40}, 20^{30}?

Question 3

25 married couples sit at a large round table so that no husband and wife sit adjacent to each other.
In how many different ways can this be done?

Hints: look up exclusion inclusion, try smaller numbers first.

Question 4

Write $2(a^2+b^2+c^2+d^2)(p^2+q^2+r^2+s^2) - 2(ap+bq+cr+ds)^2$ as a sum of six squares.

Question 5

Find the positive square root of the repeating decimal 0.012345679012345679....

Question 6

Justify Euler's formula:

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}.$$

Hint: look up the Riemann zeta function.

Question 7

Find all positive integers n, such that $S_n = \sum_{j=1}^{n} j^2$ is a perfect square.

Hint: we observed in class that $n = 1$ and $n = 24$ work.
Now try to exclude all other possibilities, looking at the factorization of the exact sum into squares.

Question 8

Find a monic polynomial over the integers (monic means that the leading coefficient is one) with $\sqrt{2} + \sqrt{3} + \sqrt{5}$ as a root.

Hint: what other roots must there be?

Question 9

Let a_0, a_1, a_2, ..., a_n be real numbers, such that:

$$a_0 = -\frac{a_1}{2} - \frac{a_2}{3} - \frac{a_3}{4} - \cdots - \frac{a_n}{n+1}$$

Prove that the polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots a_n x^n$ has at least one real root.

Hint: integrate to get the required formula.

Question 10

Evaluate the following limit, for $a \neq 1$ a given positive real number:

$$\lim_{x \to \infty} \left[\frac{a^x - 1}{x(a-1)} \right]^{\frac{1}{x}}$$

Hint: reduce either to the case $0 < a < 1$, or to the case $a > 1$ and take logarithms.

Question 11

For which integers a does the polynomial $x^2 + x + a$ exactly divide the polynomial $x^{13} + x - 90$?

Hint: solve the quadratic and show that the roots raised to the thirteenth power are usually large in size.

Question 12

Write $\left(1 + \sqrt{3}\right)^{2n} = a_n - b_n$, where n and a_n are positive integers and $0 < b_n < 1$.
Prove that 2^{n+1} exactly divides a_n.

Hint: consider the number $\left(1 - \sqrt{3}\right)^{2n}$.

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Second problem set: for discussion 9/18/8

Question 13

In how many ways can a 2 by n rectangle be filled by n rectangles each of which is 2 by 1?

Question 14

Find simple formulas for the following sums:

- $A = \sum_{k=1}^{n} k(k+1)(k+2)(k+3)(k+4)(k+5)$
- $B = \sum_{k=1}^{n}(2k-1)^2$.
- $C = \sum_{k=1}^{n}(-1)^{k-1}(2k-1)^3$.

Question 15

Find the smallest x, such that the following equation has a solution:

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{z^2}.$$

Here each of x, y and z is a positive integer.
Hint: Pythagoras.

Question 16

Find a formula for the perimeter of a regular $3(2^n)$-gon inscribed in a circle of unit radius, using only integers, addition, subtraction and square roots in your formula.
Hint: Euler's formula.

Question 17

Let n distinguishable balls be given and n jars.
In how many different ways can the balls be placed in the jars (a jar may contain from 0 to n balls)?

Question 18

Prove that for any integers x and y, 17 divides $4x^2 + 12xy + 9y^2$ if and only if 17 divides $9x + 5y$.

Question 19

Prove that no integers x and y obey the equation:

$$x^2 + 3xy - 2y^2 = 122.$$

Question 20

Find with proof all positive integer solutions of the equation:

$$x^y = y^x.$$

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Third problem set: for discussion 9/25/8

Question 21

Find, with proof, the number of ways that the number 2008 can be written as a sum of four non-negative integer squares.
Also come up with your own problem involving the number 2008.

Question 22

For each positive integer n, find the number of binomial coefficients $\binom{n}{k}$ for $0 \leq k \leq n$, that are:

- even;
  hint: consider $(1+x)^{2^n}$ mod 2;
- exactly divisible by three;
  hint: consider $(1+x)^{3^n}$ mod 3;
- exactly divisible by 343.

Question 23

Every point of the Euclidean plane is colored by exactly one of two colors. Prove that there exist two points exactly one unit apart, each with the same color.

Question 24

An Eulerian circuit in a graph is a circuit containing all edges of a graph. Prove that a graph has an Eulerian circuit iff each vertex of the graph has an even number of edges emerging from the vertex.

Question 25

Which is larger π^e or e^π?
Explain your answer.

Question 26

Show, with proof, that:

$$\int_0^1 x^x dx = -\sum_{k=1}^{\infty}(-k)^{-k}.$$

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Fourth problem set: for discussion 10/2/8

Question 27

Without using a calculator, factorize the number 999973.

Question 28

Prove that at least three people in a crowd of one thousand have the same birthday.
Ignoring leap years, what is the probability that at least four people have the same birthday?

Question 29

For any $(x, y, z) \in \mathbb{R}^3$, let $f(x, y, z) = x^2 + y^2 + z^2 + xy + yz + zx$.
Find, with proof, the volume integral:

$$\mathcal{J} = \int_{\mathbb{R}^3} e^{-f} dx dy dz.$$

Question 30

For x real, let $f(x) = \frac{x}{e^x - 1}$, for $x \neq 0$ and $f(0) = 1$.
Prove that $f(x)$ is real analytic.
Also prove that the Taylor series of $f(x)$, based at the origin, is even after the first two terms.

Question 31

Prove or disprove that the function $g(x) = \frac{x}{\sin x}$ is convex on the interval $(0, \pi)$.

Question 32

Let a function $f(x, y)$ be given by the formula, valid for real x and y, with $|x| \leq \sqrt{2}$ and $y \neq 0$:

$$f(x, y) = (x - y)^2 + \left(\sqrt{2 - x^2} - \frac{9}{y}\right)^2 .$$

Find the local and global maxima and minima of the function f, if these exist. If one does not exist, prove it.

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Fifth problem set: for discussion 10/9/8

Question 33

Let $y = f(x)$ solve the differential equation:

$$y'' - 2y' + y = 2e^x.$$

- If $f(x) > 0$ for all real x, must $f'(x) > 0$ for all real x. Explain.
- If $f'(x) > 0$ for all real x, must $f(x) > 0$ for all real x. Explain.

Question 34

The sequence of digits

$$123456789101112131415161718192021\ldots$$

is obtained by writing the positive integers in order. If the 10^n-th digit in this expansion occurs in the part of the sequence, where m-digit numbers are listed, define $f(n) = m$. Find, with proof $f(2008)$.

Question 35

Let $\mathbb{F}$ be a finite field not of characteristic two, with p elements.
Let q be the number of points on the "circle" $\{(x, y) \in \mathbb{F} \times \mathbb{F} : x^2 + y^2 = 1\}$.
Let r be the number of solutions in $\mathbb{F}$ of the equation $x^2 + 1 = 0$.
Prove that $q + r = p + 1$.

Question 36

Evaluate the following integral"

$$\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}} dx.$$

Question 37

Determine the units digit of the following number:

$$\frac{10^{20000}}{10^{100}+3}.$$

Question 38

Find the range of the function $x^3 - 3x$ on the set of all real numbers x such that $x^4 + 36 \leq 13x^2$.

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Sixth problem set: for discussion 10/23/8

Question 39

A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$.
If $f(x) = e^{x^2}$, determine, with proof, whether there exists a non-empty open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

Question 40

Determine, with proof, the set of real numbers x for which the following sum converges:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \csc \frac{1}{n} - 1 \right)^{x}.$$

Question 41

- If every point of the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart?
- If every point of the plane is painted one of nine colors, do there necessarily exist two points of the same color exactly one inch apart?

Question 42

Prove that there exists a *unique* function f from the set $\mathbb{R}^+$ of positive real numbers to $\mathbb{R}^+$ such that

$$f(f(x)) = 6x - f(x)$$

and

$$f(x) > 0$$

for all $x > 0$.

Question 43

If a linear transformation A on an n-dimensional vector space has $n+1$ eigenvectors such that any n of them are linearly independent, does it follow that A is a scalar multiple of the identity? Prove your answer.

Question 44

A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, z positive integers.

Question 45

Prove or disprove:
If x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

Question 46

How many primes among the positive integers, written as usual in base ten, are alternating 1's and 0's, beginning and ending with 1?

Question 47

For every n in the set $\mathbb{N} = \{1, 2, \dots\}$ of positive integers, let r_n be the minimum value of $|c - d\sqrt{3}|$ for all nonnegative integers c and d with $c+d = n$.
Find, with proof, the smallest positive real number g with $r_n \leq g$ for all $n \in \mathbb{N}$.

Question 48

Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty} (a_n)^{\frac{n}{(n+1)}}$.

Question 49

Prove that there exist an infinite number of ordered pairs (a, b) of integers such that for every positive integer t, the number $at+b$ is a triangular number if and only if t is a triangular number.
(The triangular numbers are the $t_n = \dfrac{n(n+1)}{2}$ with n in $\{0, 1, 2, \dots\}$.)

Question 50

For positive integers n, let M_n be the $(2n+1)$ by $(2n+1)$ skew-symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1. Find, with proof, the rank of M_n.
(According to one definition, the rank of a matrix is the largest k such that there is a $k \times k$ submatrix with nonzero determinant.)

One may note that

$$M_1 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

Question 51

Evaluate the following integral:

$$\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\, dy\, dx.$$

Here a and b are positive real numbers.

Question 52

Prove that for z a complex number, such that $11z^{10} + 10iz^9 + 10iz - 11 = 0$, then $|z| = 1$.

Question 53

If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is α?

An honest coin is one for which the probability of heads and the probability of tails are both $\frac{1}{2}$.

A game is finite if with probability 1 it must end in a finite number of moves.

Question 54

Let $\alpha = 1 + a_1 x + a_2 x^2 + \cdots$ be a formal power series with coefficients in the field $\mathbb{Z}_2$ of two elements. Let

$$a_n = \begin{cases} 1 & \text{if every block of zeros in the binary expansion of } n \text{ has an even number of zeros in the block} \\ 0 & \text{otherwise.} \end{cases}$$

For example, $a_{36} = 1$ because $36 = 100100_2$ and $a_{20} = 0$ because $20 = 10100_2$.

Prove that $\alpha^3 + x\alpha + 1 = 0$.

Question 55

A dart, thrown at random, hits a square target.

Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge.

Express your answer in the form $\dfrac{a\sqrt{b}+c}{d}$, where a, b, c, d are integers.

Question 56

Let $\mathbb{S}$ be a non-empty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$).

Assume that for every a in $\mathbb{S}$ the set $\{a^n : n = 1, 2, 3, \ldots\}$ is finite.

Must $\mathbb{S}$ be a group?

Question 57

Let f be a function on $[0, \infty)$, differentiable and satisfying, for any $x > 0$, the relation:

$$f'(x) = -3f(x) + 6f(2x).$$

Assume that $|f(x)| \leq e^{-\sqrt{x}}$ for $x \geq 0$.
For n a non-negative integer, define

$$\mu_n = \int_0^\infty x^n f(x)\, dx.$$

a) Express μ_n in terms of μ_0.

b) Prove that the sequence $\{\frac{3^n \mu_n}{n!}\}$ always converges, and that the limit is 0 only if $\mu_0 = 0$.

Question 58

Can a countably infinite set have an uncountable collection of non-empty subsets such that the intersection of any two of them is finite?

Question 59

Label the vertices of a trapezoid $\mathcal{T}$ (quadrilateral with two parallel sides) inscribed in the unit circle as A, B, C, D so that AB is parallel to CD and A, B, C, D are in counterclockwise order.

Let s_1, s_2, and d denote the lengths of the line segments AB, CD, and OE, where E is the point of intersection of the diagonals of $\mathcal{T}$, and O is the center of the circle.

Determine the least upper bound of the quantity $\dfrac{s_1 - s_2}{d}$ over all such $\mathcal{T}$ for which $d \neq 0$, and describe all cases, if any, in which it is attained.

Question 60

Let $(x_1, x_2, \ldots x_n)$ be a point chosen at random from the n-dimensional region defined by $0 < x_1 < x_2 < \cdots < x_n < 1$.
Let f be a continuous function on $[0, 1]$ with $f(1) = 0$.
Set $x_0 = 0$ and $x_{n+1} = 1$.
Show that the expected value of the Riemann sum

$$\sum_{i=0}^{n} (x_{i+1} - x_i) f(x_{i+1})$$

is $\int_0^1 f(t)P(t)\,dt$, where P is a polynomial of degree n, independent of f, with $0 \leq P(t) \leq 1$ for $0 \leq t \leq 1$.

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Seventh problem set: for discussion 11/13/8

Question 61

The rectangle with vertices (0, 0), (0, 3), (2, 0) and (2, 3) in the plane is rotated clockwise through a right angle about the point (2, 0), then about (5, 0), then about (7, 0), and finally about (10, 0).
The net effect is to translate it a distance 10 along the x-axis.
The point initially at (1, 1) traces out a curve.
Find the area under this curve (in other words, the area of the region bounded by the curve, the x-axis and the lines parallel to the y-axis through (1, 0) and (11, 0)).

Question 62

A and B are real unequal $n \times n$ matrices satisfying $A^3 = B^3$ and $A^2B = B^2A$.
Can we choose A and B, so that $A^2 + B^2$ is invertible?

Question 63

$P(x)$ is a polynomial of degree $n \geq 2$, with real coefficients, such that:

- it has n unequal real roots,
- for each pair of adjacent roots a and b the derivative $P'(x)$ vanishes at $x = \dfrac{a+b}{2}$.

Find all possible $P(x)$.

Question 64

Can we find an (infinite) sequence of disks in the Euclidean plane such that:

- their centers have no (finite) limit point in the plane;
- the total area of the disks is finite;
- every line in the plane intersects at least one of the disks?

Question 65

Let $f(z) = \int_0^z \sqrt{x^4 + (z - z^2)^2}dx$.

Find the maximum value of $f(z)$ in the range $0 \leq z \leq 1$.

Question 66

Can we find N such that all $m \times n$ rectangles with $m, n > N$ can be tiled with 4×6 and 5×7 rectangles?

Question 67

An n-sum of type 1 is a finite sequence of positive integers $a_1, a_2, \ldots, a_r$ such that:

- $a_1 + a_2 + ...a_r = n$;
- $a_1 > a_2 + a_3, a_2 > a_3 + a_4, \ldots, a_{r-2} > a_{r-1} + a_r$, and $a_{r-1} > a_r$.

For example, there are five 7-sums of type 1, namely:

$$7; 6, 1; 5, 2; 4, 3; 4, 2, 1.$$

An n-sum of type 2 is a finite sequence of positive integers, $b_1, b_2, \ldots, b_s$, such that:

- $b_1 + b_2 + ... + b_s = n$;
- $b_1 \geq b_2 \geq b_3 \cdots \geq b_s$;
- each b_i is in the sequence $1, 2, 4, \ldots, g_j, \ldots$, defined by the recursion $g_1 = 1, \quad g_2 = 2, \quad g_j = g_{j-1} + g_{j-2} + 1$;
- if $b_1 = g_k$, then $1, 2, 4, ..., g_k$ is a subsequence.

For example, there are five 7-sums of type 2, namely:

$$4, 2, 1; 2, 2, 2, 1; 2, 2, 1, 1, 1; 2, 1, 1, 1, 1, 1; 1, 1, 1, 1, 1, 1, 1.$$

Prove that for $n \geq 1$, the number of type 1 and type 2 n-sums is the same.

Question 68

For positive integers n define $d(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$.
Given a positive integer b_0, define a sequence b_i by taking $b_{k+1} = b_k + d(b_k)$.
For what b_0 do we have b_j constant for sufficiently large j?

Question 69

$\mathbb{R}$ is the real line.
$f, g : \mathbb{R} \to \mathbb{R}$ are non-constant, differentiable functions satisfying:

- $f(x+y) = f(x)f(y) - g(x)g(y)$, for all $x, y \in \mathbb{R}$;
- $g(x+y) = f(x)g(y) + g(x)f(y)$, for all $x, y \in \mathbb{R}$;
- $f'(0) = 0$.

Prove that $(f(x))^2 + (g(x))^2 = 1$, for all $x \in \mathbb{R}$.

Question 70

p is an odd prime.
Prove that:

$$\sum_{n=1}^{p} \binom{p}{n} \binom{p+n}{n} = 2^p \mod p^2.$$

Question 71

p is an odd prime.
How many residues mod p are both squares and squares plus one?

Question 72

Let a and b be positive numbers.
Find the largest number c in terms of a and b, such that for all x with $0 \leq x \leq c$ and for all α with $0 < \alpha < 1$, we have:

$$a^\alpha b^{1-\alpha} \sinh(x) \leq a \sinh(\alpha x) + b \sinh(x(1-\alpha)).$$

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Eighth problem set: for discussion 11/20/8

Question 73

$\mathbb{S}$ is a set of real numbers which is closed under multiplication.
Suppose that $\mathbb{S} = \mathbb{A} \cup \mathbb{B}$ where $\mathbb{A} \cap \mathbb{B} = \phi$.
Suppose also that, if $a, b, c \in \mathbb{A}$, then $abc \in \mathbb{A}$.
Similarly, suppose that, if $p, q, r \in \mathbb{B}$, then $pqr \in \mathbb{B}$.
Show that at least one of the sets $\mathbb{A}$ and $\mathbb{B}$ is closed under multiplication.

Question 74

For what positive reals α and β does the following integral $\mathbb{J}$ converge:

$$\mathbb{J} = \int_{\beta}^{\infty} \left(\sqrt{\sqrt{x+\alpha} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-\beta}} \right) \, dx.$$

Question 75

Numbers a, b and c each have nine digits when written in base ten.
Each of the nine numbers formed from a by replacing one of its digits by the corresponding digit of $b$is divisible by seven.
Similarly, each of the nine numbers formed from b by replacing one of its digits by the corresponding digit of c is divisible by seven.
Show that each of the nine differences between corresponding digits of a and c is divisible by seven.

Question 76

n integers totalling $n-1$ are arranged in a circle.
Prove that we choose one of the integers x_1, so that the other integers going around the circle are, in order, x_2, $x_3, \dots x_n$ and we have $\sum_{j=1}^{k} x_j \leq k-1$ for $k = 1, 2, \dots, n$.

Question 77

Let $x_i : \mathbb{R} \to \mathbb{R}$ be differentiable for $i = 1, 2, \ldots, n$ and satisfy the differential system, for some constants $a_{ij} \geq 0$:

$$x_i' = \sum_{j=1}^{n} a_{ij} x_j.$$

Assume that $\lim_{t\to\infty} x_i(t) = 0$.
Can the functions x_i be linearly independent?

Question 78

Each of the n triples (r_i, s_i, t_i) is a randomly chosen permutation of $(1, 2, 3)$ and each triple is chosen independently.
Put $R = \sum_{i=1}^{n} r_i$,$S = \sum_{i=1}^{n} s_i$, $T = \sum_{i=1}^{n} t_i$.

Let p be the probability that each of the three sums R, S and T equals $2n$, and let q be the probability that the three sums are $2n-1, 2n, 2n+1$ in some order.
Show that for some $n \geq 1995$, we have $4p \leq q$.

Question 79

Let $\mathbb{X}$ be a set with nine elements.
Given a partition π of $\mathbb{X}$, let $\pi(h)$ be the number of elements in the part containing h.
Given any two partitions π_1 and π_2 of $\mathbb{X}$, show that we can find $h \neq k$ such that $\pi_1(h) = \pi_1(k)$ and $\pi_2(h) = \pi_2(k)$.

Question 80

An ellipse with semi-axes a and b rolls without slipping on the curve with equation $y = c \sin\left(\dfrac{x}{a}\right)$ and completes one revolution in one period of the sine curve.
What conditions do a, b, c satisfy?

Question 81

For each positive integer k with n^2 decimal digits (and leading digit non-zero), let $d(k)$ be the determinant of the matrix formed by writing the digits in order across the rows (so if k has decimal form $a_1 a_2 \ldots a_n$, then the matrix has elements $b_{ij} = a_{n(i-1)+j}$.
Find $f(n) = \sum d(k)$, where the sum is taken over all $9(10^{n^2-1})$ such integers.

Question 82

Let y be the following number:

$$y = \left(2207 - \frac{1}{\left(2207 - \frac{1}{\left(2207 - \frac{1}{(2207 - \ldots)}\right)} \right)} \right)^{\frac{1}{8}}.$$

Write y in the form $y = a + \sqrt{b}$, where a and b are positive rational numbers.

Question 83

A game starts with four heaps, containing $3, 4, 5$ and 6 items respectively.
The two players move alternately.
A player may take a complete heap of two or three items or take one item from a heap provided that leaves more than one item in that heap.
The player who takes the last item wins.
Give a winning strategy for the first or second player.

Question 84

Let $\mathbb{N}$ be the positive integers.
For any $\alpha > 0$, define $\mathbb{S}_\alpha = \{[n\alpha] : n \in \mathbb{N}\}$.
Prove that we cannot find positive real numbers α, β and γ, such that we have the decomposition $\mathbb{N} = \mathbb{S}_\alpha \cup \mathbb{S}_\beta \cup \mathbb{S}_\gamma$ and $\mathbb{S}_\alpha$, $\mathbb{S}_\beta$ and $\mathbb{S}_\gamma$ are (pairwise) disjoint.