Differential Equations
Homework 9 Solutions, 3/28/7

Question 1
Classify the nature of the equilibria for the differential equation of a pendulum: \( x'' = -\sin(x) \) (begin by writing this a first order system).
Also interpret this nature physically, using the fact that \( x \) is the radian angle that that the pendulum makes with the downward vertical.

We write \( x' = y \) and then \( y' = x'' = -\sin(x) \), so the equivalent first order system is:

\[
(x', y') = (y, -\sin(x)) = (f(x, y), g(x, y)),
\]

\[
f(x, y) = y, \quad g(x, y) = -\sin(x).
\]

The equilibria are where \( f(x, y) \) and \( g(x, y) \) both vanish, so where \( y \) and \( \sin(x) \) both vanish, so at the points \( (n\pi, 0) \), for \( n \) an integer.

The Jacobian \( J(f, g) \) is:

\[
J(f, g) = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -\cos(x) & 0 \end{vmatrix}
\]

The eigen-value equation is:

\[
0 = \det \begin{vmatrix} -\lambda & 1 \\ -\cos(x) & -\lambda \end{vmatrix} = \lambda^2 + \cos(x).
\]

We need to evaluate this at the equilibria \( (n\pi, 0) \).

- When \( n \) is even, we have \( n\pi \) an even multiple of \( \pi \), so \( \cos(\pi) = 1 \) and the eigen-value equation is \( \lambda^2 + 1 = 0 \), so the roots are \( \lambda = \pm\sqrt{-1} = \pm i \).
The linearized system is \( u' = v \) and \( v' = -u \), which we recognize as simple harmonic motion, with circular trajectories about the equilibrium point.
However this is non-generic, so we cannot assert that the actual pendulum behaves this way.
• When $n$ is odd, we have $n\pi$ an odd multiple of $\pi$, so $\cos(\pi) = -1$ and the eigen-value equation is $\lambda^2 - 1 = 0$, so the roots are $\lambda = \pm 1$.

This is a standard saddle, which is generic, so the actual pendulum also behaves like a saddle near these equilibria.

Physically, for the pendulum the variable $x$ represents the radian angle that the pendulum makes with the downward vertical, whereas the variable $y$ represents the angular velocity of the pendulum. The equilibria occur when the pendulum is vertical. This happens in two ways:

• When $(x, y) = (n\pi, 0)$, with $n$ odd, the pendulum is at rest upside down and the slightest displacement sends it rotating downwards. This is the saddle case, where the trajectories rapidly deviate from the equilibrium. These equilibria are unstable.

• When $(x, y) = (n\pi, 0)$, with $n$ even, the pendulum is at rest in the downward position. A small displacement leads to a periodic motion (hence: clocks!), so in fact the trajectories near these equilibria are closed orbits, like circles. These equilibria are called circularly stable.
Question 2

Classify the nature of the equilibria for the following system:

\[ x' = 4x(3 - x) - 3xy, \quad y' = y(1 - y) - xy. \]

Also explain why a solution that starts in the first quadrant must remain there for all time.

We have \( x' = f(x, y), \quad y' = g(x, y), \) where:

\[
\begin{align*}
  f(x, y) &= 4x(3 - x) - 3xy = 12x - 4x^2 - 3xy = x(12 - 4x - 3y), \\
  g(x, y) &= y(1 - y) - xy = y - y^2 - xy = y(1 - y - x).
\end{align*}
\]

- If \( x = 0 \), then \( f(x, y) = 0 \) and \( g(x, y) = y(1 - y) \), which vanishes when \( y = 0 \) or \( y = 1 \), giving \((0, 0)\) and \((0, 1)\) as equilibria.

- If \( y = 0 \), then \( g(x, y) = 0 \) and \( f(x, y) = x(12 - 4x) \), which vanishes when \( x = 0 \) or \( 4x = 12 \), so \( x = 3 \), giving \((0, 0)\) and \((3, 0)\) as equilibria.

- If \( x \) and \( y \) are both non-zero, then \( f(x, y) = 0 = g(x, y) \) entails that \( 12 - 4x - 3y = 0 \) and \( 1 - y - x = 0 \), so \( y = 1 - x \) and \( 0 = 12 - 4x - 3(1 - x) = 9 - x \), so \( x = 9 \) and then \( y = 1 - 9 = -8 \).

So there are four equilibria at \((0, 0)\), \((0, 1)\), \((3, 0)\) and \((9, -8)\).

The Jacobian matrix is:

\[
J(f, g) = \begin{vmatrix}
  12 - 8x - 3y & -3x \\
-3 & 1 - 2y - x
\end{vmatrix}
\]

- When \((x, y) = (0, 0)\), we get:

\[
J(f, g) = \begin{vmatrix}
  12 & 0 \\
0 & 1
\end{vmatrix}
\]

Since this matrix is diagonal, its eigen-values are 12 and 1, so we have a (generic) nodal source.
• When \((x, y) = (0, 1)\), we get:

\[
J(f, g) = \begin{bmatrix}
9 & 0 \\
-1 & -1
\end{bmatrix}
\]

Since this matrix is lower triangular, its eigen-values are 9 and \(-1\), so we have a (generic) saddle.

• When \((x, y) = (3, 0)\), we get:

\[
J(f, g) = \begin{bmatrix}
-12 & -9 \\
0 & -2
\end{bmatrix}
\]

Since this matrix is upper triangular, its eigen-values are \(-12\) and \(-2\), so we have a (generic) nodal sink.

• When \((x, y) = (9, -8)\), we get:

\[
J(f, g) = \begin{bmatrix}
-36 & -27 \\
8 & 8
\end{bmatrix}
\]

The eigen-value equation for this system is:

\[
0 = (-36 - \lambda)(8 - \lambda) - (-27)8 = \lambda^2 + 28\lambda - 72 = (\lambda + 14)^2 - 268.
\]

The roots are \(-14 \pm \sqrt{268}\).
But \(\sqrt{268} > 14\), since \(14^2 = 196\), so one root is positive and one negative, so the equilibrium is a saddle.

When \(x = 0\), the direction field is \((0, y(1 - y))\), which points along the \(y\)-axis and any solution beginning on the \(y\)-axis stays there.
Also when \(y = 0\), the direction field is \((4x(3 - x), 0)\), which points along the \(x\)-axis and any solution beginning on the \(x\)-axis stays there.
So no solution can cross any axis. In particular any solution starting inside the first quadrant stays there, any solution starting on the positive \(x\)-axis stays there and any solution starting on the positive \(y\)-axis stays there.
Finally any solution starting at the origin stays there.
So any solution starting inside or on the first quadrant stays there.
Question 3

Let \( f(t) = \sin(3t) \).

Determine the Laplace transforms of the following functions:

- \( f(t) \)
  
  We know that \( \sin(at) \) has Laplace transform \( \frac{a}{s^2 + a^2} \).
  
  Here \( a = 3 \), so \( \mathcal{L}(f) = \frac{3}{s^2 + 9} \).

- \( f(2t) \)
  
  We have \( f(2t) = \sin(6t) \), so \( \mathcal{L}(f) = \frac{6}{s^2 + 36} \).
  
  Alternatively, we have, for \( k > 0 \):

  \[
  \mathcal{L}(f(kt))(s) = \int_0^\infty e^{-st} f(kt)dt = \int_0^\infty e^{-s \left( \frac{u}{k} \right)} f(u)du \cdot \frac{1}{k} = \frac{1}{k} \mathcal{L}(f(t)) \left( \frac{s}{k} \right) = \mathcal{L}(f(t)) \left( \frac{s}{k} \right).
  \]

  Here we made the substitution \( t = \frac{u}{k} \), \( dt = \frac{du}{k} \).

  Note that since \( k > 0 \), the integration interval is still \([0, \infty)\).

  In particular for the case \( k = 2 \), and for \( f(t) = \sin(3t) \), we have:

  \[
  \mathcal{L}(f(2t)) = \mathcal{L}(f) \left( \frac{s}{2} \right) = \frac{1}{2} \left( \frac{3}{\left( \frac{3}{2} \right)^2 + 9} \right) = \frac{1}{2} \left( \frac{3}{\frac{9}{4} + 9} \right) = \frac{3}{\frac{27}{4} + 9} = \frac{12}{s^2 + 36} = \frac{6}{s^2 + 36}.
  \]
• $f'(t)$

Since $f(t) = \sin(3t)$, we have $f'(t) = 3\cos(3t)$.

We know that $\cos(at)$ has Laplace transform $\frac{s}{s^2 + a^2}$.

Here $a = 3$, so we get:

$$
\mathcal{L}(f') = 3\mathcal{L}(\cos(at)) = 3\left(\frac{s}{s^2 + 9}\right) = \frac{3s}{s^2 + 9}.
$$

Alternatively, we use the formula:

$$
\mathcal{L}(f')(s) = s\mathcal{L}(f) - f(0).
$$

So here, since $f(0) = \sin(0) = 0$, we get:

$$
\mathcal{L}(f')(s) = s\mathcal{L}(f) = \frac{3s}{s^2 + 9}.
$$

• $e^{-t}f(t)$

We use the formula:

$$
\mathcal{L}(e^{kt}f(t))(s) = \mathcal{L}(f)(s - k).
$$

Here $k = -1$, so we have:

$$
\mathcal{L}(e^{-t}f(t))(s) = \frac{3}{(s - (-1))^2 + 9} = \frac{3}{(s + 1)^2 + 9} = \frac{3}{s^2 + 2s + 10}.
$$

• $t^2f(t)$

We use the formula, for $k$ a positive integer:

$$
\mathcal{L}(t^k f(t))(s) = \left(-\frac{d}{ds}\right)^k \mathcal{L}(f)(s).
$$

Here $k = 2$, so we have:

$$
\mathcal{L}(t^2 f(t))(s) = \frac{d^2}{ds^2} \left(\frac{3}{s^2 + 9}\right) = \frac{d}{ds} \left(\frac{-6s}{(s^2 + 9)^2}\right)
$$

$$
= \frac{d}{ds}(-6s(s^2 + 9)^{-2} = -6(s^2 + 9)^{-2} - 6s(-2)(2s)(s^2 + 9)^{-3}
$$

$$
= -6(s^2 + 9)^{-2} + 24s^2(s^2 + 9)^{-3} = (s^2 + 9)^{-3}(-6(s^2 + 9) + 24s^2) = \frac{18(s^2 - 3)}{(s^2 + 9)^3}.
$$
Question 4

Let a function $f(t)$ have Laplace transform the function $g(s) = \frac{3s + 10}{s^2 + 6s + 8}$.
Find the function $f$ and determine the Laplace transform of $f'(t)$.

We use partial fractions:

$$\frac{3s + 10}{s^2 + 6s + 8} = \frac{3s + 10}{(s + 2)(s + 4)} = \frac{A}{s + 2} + \frac{B}{s + 4}.$$  

Multiplying both sides by $(s + 2)(s + 4)$ we get:

$$3s + 10 = A(s + 4) + B(s + 2).$$

Putting $s = -2$, we get: $4 = 2A$, so $A = 2$.

Putting $s = -4$, we get: $-2 = -2B$, so $B = 1$.

So we get:

$$\mathcal{L}(f(t))(s) = \frac{3s + 10}{s^2 + 6s + 8} = \frac{2}{s + 2} + \frac{1}{s + 4}.$$  

Since $e^{at}$, for a constant, has Laplace transform $\frac{1}{s - a}$ and since the Laplace transform is linear, we see that the required solution is:

$$f(t) = 2e^{-2t} + e^{-4t}.$$  

Then $f(0) = 3$, so we get:

$$\mathcal{L}(f'(t))(s) = s\mathcal{L}(f(t))(s) - f(0) = \frac{s(3s + 10)}{s^2 + 6s + 8} - 3$$

$$= \frac{3s^2 + 10s - 3(s^2 + 6s + 8)}{s^2 + 6s + 8} = \frac{-8s - 24}{s^2 + 6s + 8}.$$  

Alternatively, we have:

$$f'(t) = \frac{d}{dt} (2e^{-2t} + e^{-4t}) = -4e^{-2t} - 4e^{-4t},$$

$$\mathcal{L}(f'(t))(s) = -4\mathcal{L}(e^{-2t})(s) - 4\mathcal{L}(e^{-4t})(s)$$

$$= -\frac{4}{s + 2} - \frac{4}{s + 4} = -\frac{4(s + 2) - 4(s + 4)}{(s + 2)(s + 4)}$$

$$= \frac{-8s - 24}{s^2 + 6s + 8}.$$
Question 5

By using Laplace transforms, solve the differential equation \( y' - 5y = e^{3t} \), with the initial condition \( y(0) = 5 \).

The Laplace transform of \( y' \) is \( s\mathcal{L}(y) - y(0) = s\mathcal{L}(y) - 5 \).
So taking the Laplace transform to the differential equation, we get:

\[
\begin{align*}
\mathcal{L}(y') - 5\mathcal{L}(y) &= \mathcal{L}(e^{3t}) = \frac{1}{s-3}, \\
\mathcal{L}(y) &= \frac{1}{s-3} + \frac{1}{(s-3)(s-5)}.
\end{align*}
\]

Now we have the partial fraction decomposition:

\[
\frac{1}{(s-3)(s-5)} = \frac{A}{s-3} + \frac{B}{s-5}.
\]

Multiplying both sides by \((s-3)(s-5)\) we get:

\[
1 = A(s-5) + B(s-3).
\]

Putting \( s = 3 \), we get: \( 1 = -2A \), so \( A = -\frac{1}{2} \).

Putting \( s = 5 \), we get: \( 1 = 2B \), so \( B = \frac{1}{2} \).

So we get:

\[
\mathcal{L}(y) = \frac{5}{s-5} + \frac{1}{(s-3)(s-5)} = \frac{5}{s-5} + \frac{1}{2} \left( \frac{1}{s-5} - \frac{1}{s-3} \right) = \frac{1}{2} \left( \frac{11}{s-5} - \frac{1}{s-3} \right).
\]

Since \( e^{at} \) has Laplace transform \( \frac{1}{s-a} \) and since the Laplace transform is linear, we see that the required solution is:

\[
y = \frac{1}{2}(11e^{5t} - e^{3t}).
\]

Check \( y(0) = \frac{1}{2}(11 - 1) = 5 \) and:

\[
y' - 5y = \frac{1}{2}(55e^{5t} - 3e^{3t}) - \frac{5}{2}(11e^{5t} - e^{3t}) = e^{3t}.
\]