Differential Equations, Exam 1 Solutions, 2/9/7

Question 1

Sketch the direction field for the following logistic differential equation:

\[
\frac{dy}{dt} = \frac{y}{5} (20 - y).
\]

Give the steady state solutions.

Discuss the behavior of the solution for each of the following initial conditions, providing appropriate sketches:

- \( y(0) = -10, \)
- \( y(0) = 5, \)
- \( y(0) = 25. \)

If a solution is bounded for all time, what can we say about its maximum rate of change?

Explain your answer.

The direction field is a standard logistic field: the slopes at \( y = 20 \) and \( y = 0 \) are zero, corresponding to the steady state solutions \( y = 20 \) and \( y = 0 \), respectively.

The slopes above \( y = 20 \), or below \( y = 0 \) are negative and become more negative as one goes further from the steady states.

The slopes between \( y = 0 \) and \( y = 20 \) are positive, the maximum occurring at \( y = 10 \) and decreasing towards zero as \( y = 0 \) is approached from above, or \( y = 20 \) is approached from below.

- If a solution starts with \( y < 0 \), in particular for the case of \( y(0) = -10 \), then in finite time in the future it goes to minus infinity, whereas in the past, it approaches its horizontal asymptote \( y = 0 \) from below. Such a solution is always strictly decreasing and concave down.

- If a solution starts with \( 0 < y < 20 \), in particular for the case of \( y(0) = 5 \), then in the past it goes to the horizontal asymptote \( y = 0 \), which it approaches from above, whereas in the future, it approaches the horizontal asymptote \( y = 20 \) from below. Such a solution is always strictly increasing and is concave down when \( y \geq 10 \) and is concave up when \( y \geq 10 \).
If a solution starts with $y > 20$, in particular for the case of $y(0) = 25$, then in finite time in the past it goes to infinity, whereas in the future, it approaches its horizontal asymptote $y = 20$ from above. Such a solution is always strictly decreasing and concave up.

If a solution is bounded for all time, then either it is a steady state solution (so $y = 0$, for all time, or $y = 20$ for all time), for which the rate of change is always zero, or its initial condition lies in the range $(0, 20)$, for which the maximum rate of change occurs at its inflection point, when $y = 10$. The maximum rate of change is then $\frac{10}{5}(20 - 10) = 2(10) = 20$ units per unit time.
Question 2

A rocket approaching the planet Jupiter at a speed of 2000 kilometers per second slows down by decelerating at a constant rate of 5 meters per second per second.

- Write and solve the differential equation governing its motion.
- If the rocket is to arrive at the surface of Jupiter with zero velocity, at what distance from Jupiter, and how long before it arrives at Jupiter, should the rocket start its deceleration?

We measure distance \(x\) meters from the point at which the deceleration begins. Then the differential equation governing the motion at time \(t\) seconds is:

\[
v' = x'' = -5.
\]

Here \(v = x'\) meters per second is the velocity.

Integrating, we get:

\[
v = -5t + C.
\]

Initially, we have \(v = 2(10^6)\) meters per second, so \(C = 2(10^6)\).

So we have:

\[
v = x' = -5t + 2(10^6).
\]

Integrating again, we get:

\[
x = -\frac{5t^2}{2} + 2(10^6)t + B.
\]

Initially we have \(x = 0\), so we get \(B = 0\).

The rocket lands when \(v = 0\), so when \(5t = 2(10^6) = 20(10^5)\), so when \(t = 4(10^5)\) seconds.

Its position then is:

\[
x = -\frac{5(4(10^5))^2}{2} + 2(10^6)4(10^5)
\]

\[
= 8(10^{11}) - \frac{5(16(10^{10}))}{2} = 8(10^{11}) - 40(10^{10}) = 4(10^{11}).
\]

So the deceleration begins \(4(10^{11})\) meters, or \(4(10^8)\) kilometers or 400 million kilometers away from Jupiter’s surface and the time to landing is then \(4(10^5)\) seconds or four hundred thousand seconds or four days, fifteen hours, six minutes and forty seconds.
Question 3
Solve the following differential equations and discuss the behavior of the solutions, both forward and backward in the time \( t \):

\( \bullet \) \( t \frac{dx}{dt} + 3x = 5t^2, \quad x(1) = 4. \)

This is linear, we first put it into standard form, by dividing through by \( t \):

\[ x' + \frac{3}{t} x = 5t. \]

The integrating factor is: \( e^{\int \frac{3}{t} dt} = e^{3 \ln(t)} = e^{\ln(t^3)} = t^3. \)

Mutiplying by the integrating factor, we get:

\[ t^3 x' + 3t^2 x = 5t^4, \]

\[ (xt^3)' = 5t^4, \]

\[ xt^3 = \int 5t^4 dt = t^5 + C. \]

Putting \( t = 1 \) and \( x = 4 \), we get: \( 4 = 1 + C \), so \( C = 3 \).

So the solution is:

\[ xt^3 = t^5 + 3, \]

\[ x = t^2 + \frac{3}{t^3}. \]

The solution is defined for all positive time and goes to \( \infty \) as \( t \to 0^+ \)
and as \( t \to \infty \).

We have \( x' = 2t - 9t^{-4} \) and \( x'' = 2 + 36t^{-5} \).

So \( x'' > 0 \) for \( t > 0 \).

There is one critical point given by \( t = 3^{\frac{5}{2}} 2^{-\frac{1}{6}} = 1.350960039 \), where \( x = 5(12)^{-\frac{1}{6}} = 3.041821710. \)

So the graph decreases from infinity at \( t = 0 \), to its global minimum at \((1.350960039, 3.041821710)\) and then increases going to infinity as \( t \to \infty \) and the graph is always concave up.
\begin{itemize}
\item \( \frac{dy}{dt} = (1 - 2t)(1 + y), \quad y(0) = 0. \)
\end{itemize}

This is separable.
We separate and integrate:
\[
\int \frac{dy}{1+y} = \int (1 - 2t)dt, \\
\ln(1 + y) = t - t^2 + C.
\]
Putting \( t = 0 \) and \( y = 0 \), we get: \( \ln(1) = C \), so \( C = 0 \) and the solution is:
\[
\ln(1 + y) = t - t^2, \\
1 + y = e^{t - t^2}, \\
y = e^{t - t^2} - 1.
\]
We have \( y' = (1 - 2t)e^{t - t^2} \) and \( y'' = (-2 + (1 - 2t)^2)e^{t - t^2}. \)
The solution is defined for all time and goes to \( y = -1 \) from above as \( t \to \pm \infty \).
The graph is essentially flat for large \( t \), with a blip near the origin.
It is increasing and concave up until \( t = \frac{1}{2}(1 - \sqrt{2}) \) at the point \((-0.2071067810, -0.2211992172)\).
It then inflects and rises, concave down to its absolute maximum at \((0.5, e^{0.25} - 1) = (0.5, 0.284025417)\).
Then it decreases, still concave down until it reaches the second inflection point when \( t = \frac{1}{2}(1+\sqrt{2}) \) at the point \((1.207106781, -0.2211992172)\).
Then it inflects to concave up and decreases towards the asymptote \( y = -1 \).
The graph is symmetrical about the vertical line \( t = \frac{1}{2} \).

Note that the differential equation is also a linear differential equation, with constant solution \( y = -1 \).
The associated homogeneous equation is \( y' = (1 - 2y)y, \) with solution \( \int y^{-1}dy = \int (1 - 2t)dt, \) or \( \ln(Ay) = t - t^2, \) or \( y = Ce^{t - t^2}. \)
Adding to the particular solution gives the general solution in the form \( y = Be^{t - t^2} - 1, \) in agreement with our earlier results.
Question 4

Tank A initially contains fifty gallons of water in which are dissolved forty pounds of salt.
A salt water solution containing 0.4 pounds of salt per gallon begins to enter the tank at a rate of ten gallons per minute.
The well-mixed fluid leaves the tank through a pipe at a rate of ten gallons per minute and goes into tank B, which initially contains 100 gallons of pure water.
The well-mixed fluid in tank B is drained through another pipe also at a rate of ten gallons per minute.
Determine the amount of salt in the tanks at time $t$ minutes after the start of the process.
Also determine the ultimate amount of salt in each tank.

Let there be $x$ pounds of salt in the tank $A$ and $y$ pounds of salt in tank $B$ at time $t$ minutes.
We note that tank $A$ always contains fifty gallons of fluid, whereas tank $B$ always contains 100 gallons of fluid, since the volume of fluid leaving each tank per minute matches the volume of fluid entering each tank per minute.

- The inflow rate into tank $A$ is $10(0.4) = 4$ pounds of salt per minute.
- The outflow rate from tank $A$ is $10 \left( \frac{x}{50} \right) = \frac{x}{5}$ pounds of salt per minute.
- The inflow rate into tank $B$ is $\frac{x}{5}$ pounds of salt per minute.
- The outflow rate from tank $B$ is $10 \left( \frac{y}{100} \right) = \frac{y}{10}$ pounds of salt per minute.

So the differential system we need to solve is:

$$x' = 4 - \frac{x}{5}, \quad y' = \frac{x}{5} - \frac{y}{10}.$$ 

For $x$ the steady state solution is $0 = 4 - \frac{x}{5}$, so $x = 20$.
The associated homogeneous equation is $x' = -\frac{x}{5}$, with general solution $x = Ce^{-\frac{t}{5}}$. 


So the required solution for $x$ is:

$$x = Ce^{\frac{t}{5}} + 20.$$  

Initially $x = 40$, so we need $40 = C + 20$, so $C = 20$ and our solution for $x$ is:

$$x = 20(1 + e^{\frac{-t}{5}}).$$

Then the equation for $y$ becomes:

$$y' = \frac{x}{5} - \frac{y}{10} = 4(1 + e^{\frac{-t}{5}}) - \frac{y}{10},$$

$$y' + \frac{y}{10} = 4(1 + e^{\frac{-t}{5}}).$$

This is linear with the integration factor is: $e^{\int \frac{1}{10} dt} = e^{\frac{t}{10}}$.

So the equation becomes:

$$ye^{\frac{t}{10}} = \int 4e^{\frac{t}{10}}(1 + e^{\frac{-t}{5}})dt$$

$$= 4 \int (e^{\frac{t}{10}} + e^{\frac{-t}{5}})dt$$

$$= 40(e^{\frac{t}{10}} - e^{\frac{-t}{5}}) + D.$$  

Putting $t = 0$ and $y = 0$, we get the relation $0 = 40(1 - 1) + D$, so $D = 0$ and the required solution is:

$$ye^{\frac{t}{10}} = 40(e^{\frac{t}{10}} - e^{\frac{-t}{5}}),$$

$$y = e^{-\frac{t}{5}}40(e^{\frac{t}{10}} - e^{\frac{-t}{5}}) = 40(1 - e^{-\frac{t}{5}}).$$

So we have the solution:

$$(x, y) = (20, 40) + e^{\frac{-t}{5}}(20, -40).$$

As $t \to \infty$, the exponential terms die away and we are left with twenty pounds of salt in tank $A$ and forty pounds of salt in tank $B$. 

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Question 5

A particle of mass five kilograms falls from rest under gravity, in a medium where the frictional retarding force (measured in Newtons) is 10 times the velocity (where the velocity is measured in meters per second).

- Determine the velocity and position of the particle as a function of time $t$ seconds after the particle begins dropping.
- Approximately how long does it take for the particle to fall a distance of one kilometer?

Newton’s Second Law gives: $mv' = mg - kv$.

Here $v$ is the velocity in meters per second, measured downwards. $m = 5$ kilos is the mass, $g = 9.8$ meters per second per second is the acceleration due to gravity and $k = 10$ (the units of $k$ are kilos per second).

So we have:

$$5v' = 5(9.8) - 10v, \quad v' = 9.8 - 2v.$$

The steady state solution is $0 = 9.8 - 2v$, so $v = 4.9$.

The associated homogeneous equation is $v' = -2v$, with general solution $v = Ce^{-2t}$.

So the required solution takes the form: $v = Ce^{-2t} + 4.9$.

Putting $t = 0$, we need $v = 0$, so $0 = C + 4.9$ and $C = -4.9$.

Then we have:

$$v = v' = 4.9(1 - e^{-2t}).$$

Here $x$ meters is the distance from the initial point measured downwards.

Integrating, we get:

$$x = 4.9(t + \frac{1}{2}e^{-2t}) + B = 2.45(2t + e^{-2t}) + B.$$

Putting $t = 0$ and $x = 0$, we get: $0 = 2.45 + B$, so $B = -2.45$ and our solution is:

$$x = 2.45(2t - 1 + e^{-2t}).$$

To fall one kilometer, we need to solve the equation: $1000 = 2.45(2t-1+e^{-2t})$.

Since the limiting velocity is only 4.9 meters per second, $t$ is at least 200, so the exponential term is negligible. So we need $1000 = 2.45(2t - 1)$, which gives:

$$2t = \frac{1000}{2.45} + 1 = \frac{20000}{49} + 1 = \frac{20049}{49}, \quad t = \frac{20049}{98} = 204.5816327.$$

So it takes about 204.5816327 seconds for the particle to fall one kilometer.
Question 6

Determine the solution \( x(t) \) to the following differential equation and discuss the behavior of the solution as a function of \( t \), both backward and forward in the time variable \( t \):

\[ x'' + 2x' - 15x = 36e^{-3t}, \]

with initial conditions \( x(0) = 0, \ x'(0) = 2 \).

We put \( x = Ce^{-3t} \) to find a particular solution:

\[ x = Ce^{-3t}, \quad x' = -3Ce^{-3t}, \quad x'' = 9Ce^{-3t}, \]

\[ 0 = x'' + 2x' - 15x - 36e^{-3t} = e^{-3t}(9C - 6C - 15C - 36) = -12e^{-3t}(C + 3). \]

So \( C = -3 \) and the particular solution is \( y = -3e^{-3t} \).

The associated homogeneous equation is \( 0 = (D^2 + 2D - 15)x = (D - 3)(D + 5)x \).

The roots are 3 and \(-5\), so the general solution of the homogeneous equation is: \( x = Ae^{3t} + Be^{-5t} \).

Then the given equation has the general solution:

\[ x = Ae^{3t} + Be^{-5t} - 3e^{-3t}, \]

\[ x' = 3Ae^{3t} - 5Be^{-5t} + 9e^{-3t}. \]

Putting \( t = 0 \), we need:

\[ 0 = x(0) = A + B - 3, \]

\[ 2 = x'(0) = 3A - 5B + 9, \]

\[ 0 = 3A + 3B - 9, \]

\[ 0 = 3A - 5B + 7, \]

\[ 0 = 8B - 16, \quad B = 2, \quad A = 3 - B = 3 - 2 = 1. \]

So the required solution is:

\[ x = e^{3t} + 2e^{-5t} - 3e^{-3t}. \]

The graph of \( x \) is everywhere concave up and goes to infinity as \( t \to \pm \infty \).

Initially it is decreasing, reaching an absolute minimum at \((-0.05482960946, ?0.057224907)\).

Then it increases.

The graph of \( x \) passes through the origin and otherwise crosses the \( x \)-axis only at \( t = -0.1050299375 \).
Question 7

Determine the solution \( y(t) \) to the following differential equation and discuss the behavior of the solution as a function of \( t \), with a rough sketch of the graph of the solution:

\[
y'' + 16y = 40e^{-2t}, \quad \text{with initial conditions } y(0) = y'(0) = 0.
\]

We first find a particular solution:

\[
y = Ce^{-2t}, \quad y' = -2Ce^{-2t}, \quad y'' = 4Ce^{-2t},
\]

\[
0 = y'' + 16y - 40e^{-2t} = e^{-2t}(4C + 16C - 40) = 20e^{-2t}(C - 2).
\]

So \( C = 2 \) gives a particular solution \( y = 2e^{-2t} \).

The associated homogeneous equation is:

\[
0 = (D^2 + 16)y.
\]

The roots are \( \pm i \) giving complex solutions \( e^{\pm 4it} \).

Using Euler’s formula, we have the solution: \( y = A \cos(4t) + B \sin(4t) \).

So the general solution of the given differential equation is:

\[
y = A \cos(4t) + B \sin(4t) + 2e^{-2t}.
\]

Differentiating, we get:

\[
y' = -4A \sin(4t) + 4B \cos(4t) - 4e^{-2t}.
\]

Putting \( t = 0, \ y = 0 \) and \( y' = 0 \), we get:

\[
0 = A + 2, \quad 0 = 4B - 4.
\]

So \( A = -2 \) and \( B = 1 \), so the required solution is:

\[
y = -2 \cos(4t) + \sin(4t) + 2e^{-2t}.
\]

For \( t \) negative the solution starts out at infinity and decreases rapidly to zero at \( t = 0 \).

Then a sinusoidal oscillation sets in which for large positive \( t \) has amplitude \( \sqrt{5} = 2.236067977 \) and period \( \frac{\pi}{2} = 1.570796327 \).
Question 8

A circuit contains connected in series:

- An inductance $L$ of 3 henrys.
- A capacitance $C$ of $\frac{1}{12}$ farads.
- A resistance $R$ of 12 ohms.

The circuit is driven by a constant applied voltage of 120 volts.

Given that the initial charge in the system is zero and the initial current is zero amperes, discuss the subsequent behavior of the current and charge in the circuit, with appropriate sketches.

The differential equation for the circuit is:

$$LI' + RI + \frac{Q}{C} = V.$$ 

Here $L = 3$ henrys is the inductance, $R = 12$ ohms is the resistance and $C = \frac{1}{12}$ farads is the capacitance.

Also $I = Q'$ amperes is the current, $Q$ coulombs is the charge and $I' = Q''$.

Finally $V = 120$ volts is the driving voltage.

So here we have:

$$3Q'' + 12Q' + 12Q = 120, \quad Q'' + 4Q' + 4Q = 40.$$ 

A particular solution is the steady state solution $Q = \frac{40}{4} = 10$.

The associated homogeneous equation is:

$$0 = (D^2 + 4D + 4)Q = (D + 2)^2Q.$$ 

The repeated root is $-2$, so the general solution is $Q = (A + Bt)e^{-2t}$. 

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So the required solution of the given equation is $Q = (A + Bt)e^{-2t} + 10$.

Then $I = Q' = e^{-2t}(B - 2(A + Bt)) = e^{-2t}(B - 2A - 2Bt)$.

Putting $t = 0$, we need $0 = A + 10$ and $0 = B - 2A$.
So $A = -10$ and $B = 2A = -20$.
So we get:

$$Q = 10(1 - (2t + 1)e^{-2t}),$$
$$I = Q' = 10e^{-2t}(-2 + 2(2t + 1)) = 40te^{-2t}.$$  

Initially the current increases and is concave down, reaching its global maximum at $\left(\frac{1}{2}, 20e^{-1}\right) = (0.5, 7.357588824)$.

From then on the current decreases, switching to concave down at $(1, 40e^{-2}) = (1, 5.413411328)$ and as $t \to \infty$, the current approaches zero from below.

The charge is monotonically increasing, going from its initial value of zero asymptotically to the value 10 Coulombs, which it approaches from below.

Its graph is initially concave up, switching to concave down at $t = \frac{1}{2}$ when we have $Q = 10(1 - 2e^{-1}) = 2.642411176$. 