Homework 7 Solutions, Friday 2nd March 2007

Question 1

200 grams of a radioactive element are delivered to a hospital at noon Monday. After one day there are 160 grams of the element remaining.

- How much element remains after three days?

We have a multiplicative factor of \( \frac{160}{200} = \frac{4}{5} \) after one day, so after three days the factor is \( \left( \frac{4}{5} \right)^3 = \frac{4^3}{5^3} = \frac{64}{125} \).

So after three days we have \( 200 \left( \frac{64}{125} \right) = \frac{8(64)}{5} = \frac{512}{5} = 102.4 \) grams.

Alternatively we can proceed day by day:

\[
200 \rightarrow 160 \rightarrow \frac{4}{5}(160) = 128 \rightarrow \frac{4}{5}(128) = \frac{512}{5} = 102.4.
\]

- Write a formula for the amount of the element present \( t \) days after the delivery.

The amount \( y \) grams is given by:

\[
y = 200 \left( \frac{4}{5} \right)^t.
\]

- What is the half-life of the element?

We need:

\[
\frac{1}{2} = \left( \frac{4}{5} \right)^t, \quad t \ln \left( \frac{4}{5} \right) = \ln \left( \frac{1}{2} \right) = -\ln(2),
\]

\[
t = \frac{-\ln(2)}{\ln \left( \frac{4}{5} \right)} = \frac{-\ln(2)}{\ln(4) - \ln(5)} = \frac{\ln(2)}{\ln(5) - \ln(4)} = 3.106283724.
\]

So the half-life is 3.106283724 days, or three days, two hours, thirty-three minutes and 2.91 seconds.
• Determine the initial rate of change with respect to time of the amount of the element.

We have:

\[ y' = 200 \left(\frac{4}{5}\right)^t \ln \left(\frac{4}{5}\right), \]

\[ y'(0) = 200 \ln \left(\frac{4}{5}\right) = -44.62871026. \]

So the initial rate of decay is 44.62871026 grams per day.

• When 100 grams of the element remain, 40 grams are used in a medical procedure. When will this be?

This will be after one half-life, so at 2.33pm and 2.91 seconds on Thursday.

• The remaining 60 grams is kept until 15 grams is left, which is then used in a second medical procedure. When will this be?

This will be after an additional two half-lives, so three half-lives after noon Monday, so nine days, seven hours, thirty-nine minutes and 8.74 seconds after noon Monday, so the following week on Wednesday evening at 7.39pm and 8.74 seconds.
Question 2

Let \( y \) be defined implicitly in terms of \( x \) by the curve with equation \( \frac{(x-2)^2}{8} + \frac{(y-1)^2}{18} = 1 \).

The curve is a standard ellipse, centered at \((2, 1)\), with semi-axes \( \sqrt{8} = 2\sqrt{2} \) in the horizontal direction and \( \sqrt{18} = 3\sqrt{2} \) in the vertical direction.

Differentiating implicitly, with respect to \( x \), we get and equation for \( y' \):

\[
2\frac{(x-2)(1)}{8} + \frac{2(y-1)y'}{18} = 0.
\]

Putting \( x = 4 \) and \( y = 4 \), this becomes:

\[
\frac{2(2)}{8} + \frac{2(3)y'}{18} = 0,
\]

\[
\frac{1}{2} = \frac{y'}{3},
\]

\[
y' = \frac{3}{2}.
\]

Find the equation of the tangent and normal lines to the curve at the point \((4, 4)\) and sketch the curve and the tangent and normal lines.

So the tangent line at \((4, 4)\) has slope \( \frac{3}{2} \) so has the equation:

\[
y - 4 = \frac{3}{2}(x - 4), \quad 2y - 8 = 3x - 12, \quad 3x - 2y = 4.
\]

Also the normal line at \((4, 4)\) has slope \( -\frac{2}{3} \), so has the equation:

\[
y - 4 = -\frac{2}{3}(x - 4), \quad 3y - 12 = -2(x - 4) = -2x + 8, \quad 2x + 3y = 20.
\]
Question 3

Let \( f(x) = x^3 - 5x^2 + 3x + 9 \).

- Find the intervals on which \( y = f(x) \) is increasing and the intervals on which \( y = f(x) \) is decreasing.

We have \( f'(x) = 3x^2 - 10x + 3 = (3x - 1)(x - 3) \).

In particular \( f'(x) = 0 \) at \( x = 3 \) and at \( x = \frac{1}{3} \).

- If \( x \geq 3 \), then \( f' \geq 0 \), so \( f \) is increasing on the interval \([3, \infty)\).
- If \( \frac{1}{3} \leq x \leq 3 \), then \( f' \leq 0 \), so \( f \) is decreasing on the interval \([\frac{1}{3}, 3]\).
- If \( x \leq \frac{1}{3} \), then \( f' \geq 0 \), so \( f \) is increasing on the interval \((-\infty, \frac{1}{3}]\).

- Find the intervals on which the graph of \( y = f(x) \) is concave up and the intervals on which the graph \( y = f(x) \) is concave down.

We have \( f''(x) = 6x - 10 \).

So \( f'' \) vanishes only when \( x = \frac{5}{3} \) and \( f'' > 0 \) if \( x > \frac{5}{3} \) and \( f'' < 0 \) when \( x < \frac{5}{3} \).

So the graph of \( y = f(x) \) is concave up on the interval \([\frac{5}{3}, \infty)\) and is concave down on the interval \((-\infty, \frac{5}{3}]\).

In particular \( x = \frac{5}{3} \) is an inflection point.

Then \( y = \left(\frac{5}{3}\right)^3 - 5 \left(\frac{5}{3}\right)^2 + 3 \left(\frac{5}{3}\right) + 9 = \frac{125}{27} - \frac{125}{9} + 5 + 9 = 14 - \frac{250}{27} = \frac{128}{27} \).
Find the local minima and maxima of the function $f$. 

- When $x = \frac{1}{3}$, $f' = 0$ and $f'' < 0$, so the graph is flat and concave down so has a local maximum.

Then we have:

$$y = \left(\frac{1}{3}\right)^3 - 5\left(\frac{1}{3}\right)^2 + 3\left(\frac{1}{3}\right) + 9 = \frac{1}{27} - \frac{5}{9} + 1 + 9 = 10 - \frac{14}{27} = \frac{256}{27}.$$  

- When $x = 3$, $f' = 0$ and $f'' > 0$, so the graph is flat and concave up, so has a local minimum.

Then $y = 3^3 - 5(3^2) + 3(3) + 9 = 27 - 45 + 9 + 9 = 0$.

There are no other critical points, so there are no other local maxima and minima.

- If the domain of $f$ is taken to be the interval $[-2, 4]$ find the range of $f$ and plot its graph.

At $x = -2$, we have $y = (-2)^3 - 5(-2)^2 + 3(-2) + 9 = -8 - 20 - 6 + 9 = -25$.

From $(-2, -25)$, the graph is increasing and concave down, rising to its local maximum at $\left(\frac{1}{3}, \frac{256}{27}\right)$.

Then it decreases to its inflection point at $\left(\frac{1}{3}, \frac{128}{27}\right)$.

Then it switches to concave up, decreasing to its local minimum at $(3, 0)$.

Finally it rises, still concave up, to the end-point at $(4, 4^3 - 5(4^2) + 3(4) + 9) = (4, 64 - 80 + 12 + 9) = (4, 5)$.

The range is then $[-25, 5]$.

Finally note that $y$ vanishes at $x = 3$, so can be factored:

$$x^3 - 5x^2 + 3x + 9 = (x - 3)(x^2 - 2x + 3) = (x - 3)^2(x + 1).$$

So the crossing of the $x$-axis occurs at $(-1, 0)$.

At $x = 3$, $y$ is zero, but the axis is not crossed: the graph ”bounces off” the $x$-axis.

The intercept with the $y$-axis is at $(0, 9)$. 

5
Question 4

Find the linear approximation to the function $f(x) = \frac{1}{\sqrt{x+2}}$ based at the point with $x = 2$ and use it to estimate the value of the $f(2.1)$.
Also sketch the graphs of the function and its linear approximation and explain why the approximation is an over-estimate or an under-estimate.

We have $f(x) = (x + 2)^{-\frac{1}{2}}$, so $f(2) = (2 + 2)^{-\frac{1}{2}} = 4^{-\frac{1}{2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$.
Also $f'(x) = -\frac{1}{2}(x + 2)^{-\frac{3}{2}}$, so we get:

$$f'(2) = -\frac{1}{2}(2 + 2)^{-\frac{3}{2}} = -\frac{1}{2} \left( 4^{-\frac{3}{2}} \right) = -\frac{1}{2} \left( \frac{1}{8} \right) = -\frac{1}{16}.$$

So the tangent line to the curve $y = f(x)$ at $x = 2$ goes through the point $(2, \frac{1}{2})$ and has slope $-\frac{1}{16}$, so has the equation:

$$y - \frac{1}{2} = -\frac{1}{16}(x - 2) = -\frac{x}{16} + \frac{1}{8},$$

$$y = \frac{1}{2} - \frac{x}{16} + \frac{1}{8} = \frac{1}{16}(10 - x).$$

So the linear approximation based at $x = 2$ is $f_1(x) = \frac{1}{16}(10 - x)$.
Putting $x = 2.1$, the approximation gives:

$$f_1(2.1) = \frac{1}{16}(10 - 2.1) = \frac{7.9}{16} = 0.49375.$$

The more exact value is:

$$f(2.1) = \frac{1}{\sqrt{2.1 + 2}} = \frac{1}{\sqrt{4.1}} = 0.49386.$$

The linear estimate is a slight under-estimate because we have:

$$f'' = \frac{d}{dx} \left( -\frac{1}{2}(x + 2)^{-\frac{3}{2}} \right)$$

$$= \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) (x + 2)^{-\frac{5}{2}} = \frac{3}{4} (x + 2)^{-\frac{5}{2}} > 0.$$

So the graph of $y = f(x)$ is concave up at $x = 2$, so curves upwards from the tangent line at $x = 2$, so the true value of $y$ lies slightly above the estimated value.
Question 5

Find the derivatives of the following functions:

- \( a(x) = \sin(\sqrt{x + 3}) \)
  
  By the chain rule we have:
  
  \[
  a(x) = \sin((x + 3)^{\frac{1}{2}}),
  \]
  
  \[
  a'(x) = \cos((x + 3)^{\frac{1}{2}}) \frac{d}{dx} \left( (x + 3)^{\frac{1}{2}} \right) = \cos((x + 3)^{\frac{1}{2}}) \frac{1}{2} \left( (x + 3)^{-\frac{1}{2}} \right)
  \]
  
  \[
  = \frac{\cos(\sqrt{x + 3})}{2\sqrt{x + 3}}.
  \]

- \( b(x) = x^x \)

  We take logarithms of both sides and then differentiate implicitly:
  
  \[
  \ln(b) = x \ln(x), \quad \frac{b'}{b} = 1(\ln(x)) + x \left( \frac{1}{x} \right) = \ln(x) + 1,
  \]
  
  \[
  b' = b(\ln(x) + 1) = x^x(\ln(x) + 1).
  \]

- \( c(x) = \arctan(x^2) \).

  We have, since \( \tan(\arctan(u)) = u \):
  
  \[
  \tan(c) = x^2.
  \]

  Differentiate implicitly with respect to \( x \), using the chain rule:
  
  \[
  \sec^2(c) c' = 2x.
  \]

  But a trigonometric identity says that \( \sec^2(c) = 1 + \tan^2(c) = 1 + (x^2)^2 = 1 + x^4 \).

  So we get:
  
  \[
  c' = \frac{2x}{\sec^2(x)} = \frac{2x}{1 + x^4}.
  \]

  Alternatively, we use the derivative formula \( \arctan'(x) = \frac{1}{1 + x^2} \), together with the chain rule:
  
  \[
  c' = \frac{1}{1 + (x^2)^2}(2x) = \frac{2x}{1 + x^4}.
  \]