Homework 2 Solutions, Friday 19th January 2007

Question 1

Let \( f(x) = \frac{2-x}{x-1} \).

- Give the domain of the function \( f \).

\( f \) is defined for all real \( x \neq 1 \), so on the union of open intervals \((-\infty, 1) \cup (1, \infty)\).

- Show that the function \( f \) is one-to-one and has an inverse and find a formula for the inverse function. Also give the domain and range of \( f \) and its inverse function \( f^{-1} \).

We solve for the inverse function:

\[
y = \frac{2-x}{x-1},
\]

\[
(x - 1)y = 2 - x,
\]

\[
xy - y = 2 - x,
\]

\[
xy + x = y + 2,
\]

\[
x(y + 1) = y + 2.
\]

\[
x = \frac{y + 2}{y + 1}.
\]

Here we need to assume that \( y \neq -1 \).

But if \( y = -1 \) is an output of \( f(x) \), we have:

\[
\frac{2-x}{x-1} = -1,
\]

\[
2 - x = -1(x - 1) = 1 - x,
\]

\[
0 = 2 - x + x - 1 = 1.
\]
This is a contradiction, so \( y = -1 \) is not an output for \( f \).

So after interchanging \( x \) and \( y \), we put:

\[
f^{-1}(x) = \frac{x + 2}{x + 1}, \text{ defined for any } x \neq -1.
\]

We note at \( f^{-1}(x) \neq 1 \) for any \( x \), since the numerator and denominator of \( f^{-1} \) are never equal.

So the compositions \( f \circ f^{-1} \) and \( f^{-1} \circ f \) are well defined.

We compute these compositions:

- If \( x \neq -1 \), we have:

\[
(f \circ f^{-1})(x) = \frac{2 - f^{-1}(x)}{f^{-1}(x) - 1} = \frac{2 - \left(\frac{x + 2}{x + 1}\right)}{\left(\frac{x + 2}{x + 1}\right) - 1} = \frac{2x + 2 - x - 2}{x + 2 - x - 1} = x.
\]

- If \( x \neq 1 \), we have:

\[
(f^{-1} \circ f)(x) = \frac{f(x) + 2}{f(x) + 1} = \frac{\left(\frac{2 - x}{x - 1}\right) + 2}{\left(\frac{2 - x}{x - 1}\right) + 1} = \frac{2 - x + 2(x - 1)}{2 - x + (x - 1)} = \frac{2 - x + 2x - 2}{1} = x.
\]

So \( f \) has inverse \( f^{-1} \).

Also the range of \( f \) is the domain of \( f^{-1} \), so is all reals except for \(-1\), or the union of open intervals: \((-\infty, -1) \cup (-1, \infty)\).

Finally the range of \( f^{-1} \) is the domain of \( f \), so is all reals except for 1, or the union of open intervals: \((-\infty, 1) \cup (1, \infty)\).
• Sketch the graph of $f$ and $f^{-1}$ on the same graph.

The graphs are rectangular hyperbolas with asymptotes parallel to the axes.

- The graph of $f$ is centered at $(1, -1)$.
  Its asymptotes are $x = 1$ and $y = -1$.
  The right branch descends from infinity as $x \to 1^+$, always decreasing and concave up, going to $y = -1^+$, as $x \to \infty$.
  It crosses from the first quadrant to the fourth quadrant at $(2, 0)$.
  The left branch goes to $-1^-$ as $x \to -\infty$.
  It descends to $-\infty$ as $x \to 1^-$, always decreasing and concave down.
  It crosses from the third quadrant to the fourth quadrant at $(0, -2)$.

- The graph of $f^{-1}$ is centered at $(-1, 1)$.
  Its asymptotes are $x = -1$ and $y = 1$.
  The right branch descends from infinity as $x \to -1^+$, always decreasing and concave up, going to $y = 1^+$, as $x \to \infty$.
  It crosses from the second quadrant to the first quadrant at $(0, 2)$.
  The left branch goes to $1^-$ as $x \to -\infty$.
  It descends to $-\infty$ as $x \to -1^-$, always decreasing and concave down.
  It crosses from the second quadrant to the third quadrant at $(-2, 0)$.

Of course the two graphs are obtained from each other by reflection in the line $y = x$. 
Question 2

A radioactive element is to be used in medical procedures.

- Five grams of the element are delivered at 8am on a Monday.
- By 9am on the same day, 4.5 grams of the element remains.
- When there are three grams left, two grams are used in an operation.
- The remaining one gram is kept until only one-half of a gram remains when it is also used in another operation.

When should these operations take place? Explain your answer.

In one hour, the material decays by a factor of \( \frac{4.5}{5} = \frac{9}{10} \).

So after \( t \) hours, the material decays by a factor of \( \left( \frac{9}{10} \right)^t \).

So the amount of material left after \( t \) hours is \( P(t) = 5 \left( \frac{9}{10} \right)^t \).

There are three grams left when:

\[
3 = 5 \left( \frac{9}{10} \right)^t, \quad \frac{3}{5} = \left( \frac{9}{10} \right)^t, \\
\ln \left( \frac{3}{5} \right) = t \ln \left( \frac{9}{10} \right), \\
t = \frac{\ln(3) - \ln(5)}{\ln(9) - \ln(10)} = 4.848359189.
\]

So the first operation takes place 4.848359189 hours after 9.00am, so at fifty minutes and 54.09 seconds after 1pm.

One gram decays into half a gram when:

\[
\frac{1}{2} = 1 \left( \frac{9}{10} \right)^t, \quad \ln \left( \frac{1}{2} \right) = t \ln \left( \frac{9}{10} \right), \quad t = \frac{\ln(2)}{\ln(10) - \ln(9)} = 6.578813483.
\]

So the second operation takes place 6.578813483 hours later than the first or 11.42717267 hours after 9.00am, so at twenty-five minutes and 37.82 seconds after eight in the evening.
Question 3

Let $n$ be a positive integer.
Consider the composition $\sin \circ \sin \circ \sin \cdots \circ \sin(t)$ of the function $\sin(t)$ with itself $n$ times.
What is its domain and range? Explain your answer.
What happens as $n \to \infty$? Explain your answer.

We know from the quiz that $(\sin \circ \sin)(t)$ has range $[-\sin(1), \sin(1)]$.
This interval is inside the interval $[-1, 1]$, on which the sine function is strictly increasing.
So $(\sin \circ \sin \circ \sin)(t)$ has range $[-\sin(\sin(1)), \sin(\sin(1))]$.
Again this lies inside the interval $[-1, 1]$, so we may iterate.
Each range is then of the form $[-a_n, a_n]$, where $a_n$ is a decreasing sequence of positive numbers, such that $a_1 = \sin(1)$ and $a_n = \sin(a_{n-1})$, for any $n \geq 2$.
(The sequence is decreasing, since if $0 < x$, it is a property of the sine function that $\sin(x) < x$: in particular $a_n = \sin(a_{n-1}) < a_{n-1}$.
Computing to 50 decimal places and rounding off, we get the following sequence:

\begin{align*}
a_1 &= 0.84147, a_2 = 0.74562, a_3 = 0.67843, a_4 = 0.62757, a_5 = 0.58718, a_6 = 0.55402, \\
a_7 &= 0.52611, a_8 = 0.50217, a_9 = 0.48133, a_{10} = 0.46296, \ldots, a_{99} = 0.16967, a_{100} = 0.16886.
\end{align*}

Since the sequence $a_n$ is positive, decreasing and less than 1, a mathematical theorem then shows that $a_n$ goes to a limit $a$, where $0 \leq a < 1$.

Going to the limit in the formula $a_n = \sin(a_{n-1})$ gives the relation $a = \sin(a)$, since the function $\sin(x)$ is continuous.
But we can now use again the property of the function $\sin(x)$ that if $x > 0$, then $0 < \sin(x) < x$.
This gives the relation $\sin(a) < a$, if $a \neq 0$, a contradiction to the relation $\sin(a) = a$.
Therefore $a = 0$ and the sequence has limit zero.
Question 4

Let \( f(x) = x^2 + 2x + 3 \), defined for any real number \( x \).

Compute and interpret geometrically, using a graph of the function \( y = f(x) \), the following quantities:

- \( \frac{f(x) - f(2)}{x - 2} \)
  
  This is the chord slope \( \frac{\Delta y}{\Delta x} \) of the chord of the graph joining the points of the graph \((2, f(2))\) and \((x, f(x))\).

We have \( f(2) = 2^2 + 2(2) + 3 = 4 + 4 + 3 = 11 \), so we get:

\[
\frac{f(x) - f(2)}{x - 2} = \frac{x^2 + 2x + 3 - 11}{x - 2} = \frac{x^2 + 2x - 8}{x - 2} = \frac{(x - 2)(x + 4)}{x - 2} = x + 4.
\]

So the chord slope is \( x + 4 \).

- \( \lim_{x \to 2} \left( \frac{f(x) - f(2)}{x - 2} \right) \)
  
  This is the limit as \( x \) goes to 2 of the chord slope of the chord of the graph joining the points of the graph \((2, f(2))\) and \((x, f(x))\), so is the slope of the tangent line at \((2, f(2))\).

\[
\lim_{x \to 2} \left( \frac{f(x) - f(2)}{x - 2} \right) = \lim_{x \to 2}(x + 4) = 2 + 4 = 6.
\]

So the tangent line to the graph at \((2, 11)\) has slope 6.
\[ f(3 + h) - f(3) \]

This is the chord slope \( \frac{\Delta y}{\Delta x} \) of the chord of the graph joining the points of the graph \((3, f(3))\) and \((3 + h, f(3 + h))\).

We have \( f(3) = 3^2 + 2(3) + 3 = 9 + 6 + 3 = 18 \).

Also we have \( f(3 + h) = (3 + h)^2 + 2(3 + h) + 3 = 9 + 6h + h^2 + 6 + 2h + 3 = h^2 + 8h + 18 \).

So the chord slope is:

\[
\frac{f(3 + h) - f(3)}{h} = \frac{h^2 + 8h + 18 - 18}{h} = \frac{h^2 + 8h}{h} = h + 8.
\]

So the chord slope is \( h + 8 \).

\[ \lim_{h \to 0} \left( \frac{f(3 + h) - f(3)}{h} \right) \]

This is the limit as \( 3 + h \) goes to 3 of the chord slope of the chord of the graph joining the points of the graph \((3, f(3))\) and \((3 + h, f(3 + h))\), so is the slope of the tangent line at \((3, f(3))\).

We have:

\[
\lim_{h \to 0} \left( \frac{f(3 + h) - f(3)}{h} \right) = \lim_{h \to 0} (h + 8) = 0 + 8 = 8.
\]

So the tangent line to the graph at \((3, 11)\) has slope 8.
Question 5

The height $y$ feet of a ball thrown into the air is given by the formula, at time $t$ seconds:

$$y = 48t - 16t^2 + 6.$$ 

- Plot the graph of $y$ against $t$.

The graph is a standard parabola, curving down. When $t = 0$ we have $y = 6$. Then $y = 6$ again if:

$$6 = 48t - 16t^2 + 6,$$
$$0 = 48t - 16t^2 = 16t(t - 3).$$

So the graph returns to the level $y = 6$ at $t = 3$. Then the vertex of the parabola, lies half-way in between, so at $t = \frac{3}{2}$, when $y$ has the value:

$$y = 48 \left( \frac{3}{2} \right) - 16 \left( \frac{3}{2} \right)^2 + 6 = 72 - 36 + 6 = 42.$$

- Using your graph or otherwise, estimate the initial speed of the ball. Confirm your estimate by computing an appropriate limit. We estimate the initial speed by computing the chord slope from $t = -\frac{1}{2}$ to $t = \frac{1}{2}$:

  - When $t = -\frac{1}{2}$, we have:

$$y = 48 \left( -\frac{1}{2} \right) - 16 \left( -\frac{1}{2} \right)^2 + 6 = -24 - 4 + 6 = -22.$$

  - When $t = \frac{1}{2}$, we have:

$$y = 48 \left( \frac{1}{2} \right) - 16 \left( \frac{1}{2} \right)^2 + 6 = 24 - 4 + 6 = 26.$$

So the chord slope is:

$$\frac{\Delta y}{\Delta t} = \frac{26 - (-22)}{\frac{1}{2} - (-\frac{1}{2})} = 48.$$

So our estimate for the initial velocity is 48 feet per second.
We now compute the instantaneous velocity at (0, 6):

\[
\lim_{t \to 0} \frac{48t - 16t^2 + 6 - 6}{t - 0} = \lim_{t \to 0} \frac{48t - 16t^2}{t} = \lim_{t \to 0} 48 - 16t = 48 - 16(0) = 48.
\]

So our estimate was exactly right!

- Determine the time of the highest point on the trajectory of the ball. Confirm your value by showing that at that time, the ball has instantaneous velocity zero.

As discussed above the highest point is \( \left( \frac{3}{2}, 42 \right) \).

The instantaneous velocity then is:

\[
\lim_{t \to \frac{3}{2}} \frac{48t - 16t^2 + 6 - 42}{t - \frac{3}{2}} = 2 \lim_{t \to \frac{3}{2}} \frac{48t - 16t^2 - 36}{2t - 3}
\]

\[
= -8 \lim_{t \to \frac{3}{2}} \frac{4t^2 - 12t + 9}{2t - 3} = -8 \lim_{t \to \frac{3}{2}} \frac{(2t - 3)^2}{2t - 3}
\]

\[
= -8 \lim_{t \to \frac{3}{2}} (2t - 3) = -8 \left( 2 \left( \frac{3}{2} \right) - 3 \right) = -8(3 - 3) = 0.
\]

So the instantaneous velocity is zero at the top of the trajectory, as expected.