Analytic Geometry and Calculus I
Exam 1 Solutions 2/23/7

Question 1

Determine the derivative of each of the following functions:

- \( a(x) = 11x^3 - \frac{5}{x^2} + (x^2 + 1)\sqrt{x} + \tan(x) \)
  
  We first rewrite \( a(x) \):
  
  \[
a(x) = 11x^3 - 5x^{-2} + x^{\frac{3}{2}} + x^{\frac{1}{2}} + \tan(x).
  \]
  
  Now we differentiate using the power rule, the multiplicative constant rule, the addition and subtraction rules and that the derivative of \( \tan(x) \) is \( \sec^2(x) \):
  
  \[
a'(x) = 11(3x^2) - 5(-2x^{-3}) + \frac{5}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{-\frac{1}{2}} + \sec^2(x)
  \]
  
  \[
  = 33x^2 + 10x^{-3} + \frac{5}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{-\frac{1}{2}} + \sec^2(x).
  \]

- \( b(x) = (1 + x + \cos(x))^{10} \)
  
  We use the rules mentioned above, the generalized power rule and that the derivative of \( \cos(x) \) is \( -\sin(x) \):
  
  \[
b'(x) = 10(1 + x + \cos(x))^9(-\sin(x)).
  \]

- \( c(x) = \frac{e^{4x} - \ln(x)}{\sin(2x)} \)
  
  We use the rules mentioned above, the quotient rule and that the derivative of \( \ln(x) \) is \( x^{-1} \).
  
  Also we use the chain rule derivative formulas:
  
  \( e^u \to e^u u' \) and \( \sin(u) \to \cos(u) u' \).
  
  \[
c'(x) = \frac{(4e^{4x} - x^{-1})\sin(2x) - (e^{4x} - \ln(x))(2\cos(2x))}{\sin^2(2x)}.
  \]
Question 2

Find the following limits:

\( \lim_{x \to 2} \frac{x^2 - 7x + 10}{x^2 - x - 2} \)

We have:

\[
\lim_{x \to 2} \frac{x^2 - 7x + 10}{x^2 - x - 2} = \lim_{x \to 2} \frac{(x - 5)(x - 2)}{(x + 1)(x - 2)} \\
= \lim_{x \to 2} \frac{x - 5}{x + 1} = \frac{2 - 5}{2 + 1} = \frac{-3}{3} = -1.
\]

\( \lim_{x \to -3} \frac{|x - 3|}{9 - x^2} \)

We have:

\[
\lim_{x \to -3} \frac{|x - 3|}{9 - x^2} = \lim_{x \to -3} -\frac{(x - 3)}{9 - x^2} \\
= \lim_{x \to -3} \frac{3 - x}{(3 - x)(3 + x)} = \lim_{x \to -3} \frac{1}{3 + x} = \frac{1}{3 + 3} = \frac{1}{6}.
\]

The point here is that when \( x < 3 \), \( x - 3 \) is negative, so \( |x - 3| = -(x - 3) \).

\( \lim_{x \to -\infty} \frac{(x + 1)\sqrt{x^2 + 1}}{x(x - 5)} \)

We have:

\[
\lim_{x \to -\infty} \frac{(x + 1)\sqrt{x^2 + 1}}{x(x - 5)} = \lim_{x \to -\infty} \frac{x (1 + \frac{1}{x}) \sqrt{x^2 \sqrt{1 + \frac{1}{x^2}}}}{x(x) (1 - \frac{5}{x})} \\
= \lim_{x \to -\infty} \frac{x (1 + 0) |x| \sqrt{1 + 0}}{x(x) (1 - 0)} \\
= \lim_{x \to -\infty} \frac{|x|}{x} = \lim_{x \to -\infty} -\frac{x}{x} = \lim_{x \to -\infty} -1 = -1.
\]

The point here is that if \( x \) is negative, then \( \sqrt{x^2} = |x| = -x \).
Question 3

Find the equations of the tangent and normal lines to the curve \( y = \frac{1}{\sqrt{x+2}} \) at the point with \( x = 2 \) and sketch the curve and the lines on one graph.

When \( x = 2 \), we have \( y = \frac{1}{\sqrt{2}+2} = \frac{1}{\sqrt{4}} = \frac{1}{2} \).

Also we have \( y = (x+2)^{-\frac{3}{2}} \), so \( y' = -\frac{3}{2}(x+2)^{-\frac{5}{2}} \).

Putting \( x = 2 \), we get: \( y' = -\frac{3}{2}(2+2)^{-\frac{5}{2}} = -\frac{3}{2}(4^{-\frac{1}{2}})^{-3} = -2^{-1}2^{-3} = -2^{-4} = -\frac{1}{16} \).

So the required tangent line passes through the point of tangency \( (2, \frac{1}{2}) \) and has slope \( -\frac{1}{16} \), so has the equation:

\[
y - \frac{1}{2} = -\frac{1}{16}(x - 2),
\]

\[
16y - 8 = -(x - 2) = 2 - x,
\]

\[
x + 16y = 10.
\]

Also the required normal line passes through the point of tangency \( (2, \frac{1}{2}) \) and has slope 16, so has the equation:

\[
y - \frac{1}{2} = 16(x - 2) = 16x - 32,
\]

\[
2y - 1 = 32x - 64,
\]

\[
32x - 2y = 63.
\]

The curve steadily decreases and is concave up, with a vertical asymptote at \( x = -2 \), such that \( y \to \infty \) as \( x \to -2^+ \).

It crosses the \( y \)-axis at \((0, \frac{\sqrt{2}}{2})\) and has the \( x \)-axis as a horizontal asymptote, which it approaches from above as \( x \to \infty \).

The tangent line has slope \( -\frac{1}{16} \) and intercepts at \((0, \frac{5}{8})\) and at \((10, 0)\).

The normal line has slope 16 and intercepts at \((0, -\frac{63}{2})\) and at \((\frac{63}{32}, 0)\).
Question 4

Suppose that:
\[ f(2) = 5, \quad f'(2) = 3, \]
\[ g(2) = 2, \quad g'(2) = -3. \]

Let \( p(x) = f(x)g(x) \) and \( q(x) = f(g(x)) \).
Find the equations of the tangent and normal lines to the curves \( y = p(x) \) and \( y = q(x) \) at the points with \( x = 2 \).

- We have:
  \[ p(2) = f(2)g(2) = 5(2) = 10. \]
  Also by the product rule, we have:
  \[ p'(2) = f'(2)g(2) + f(2)g'(2) = 3(2) + 5(-3) = 6 - 15 = -9. \]
  So the tangent line to the curve \( y = p(x) \), when \( x = 2 \) passes through the point \((2, 10)\) and has slope \(-9\), so has the equation:
  \[ y - 10 = -9(x - 2), \quad y = -9x + 18 + 10 = -9x + 28. \]
  Also the normal line to the curve \( y = p(x) \), when \( x = 2 \) passes through the point \((2, 10)\) and has slope \(\frac{1}{9}\), so has the equation:
  \[ y - 10 = \frac{1}{9}(x - 2), \quad 9y - 90 = x - 2, \quad x - 9y + 88 = 0. \]

- We have:
  \[ q(2) = f(g(2)) = f(2) = 5. \]
  Also by the chain rule, we have:
  \[ q'(2) = f'(g(2))g'(2) = f'(2)g'(2) = 3(-3) = -9. \]
  So the tangent line to the curve \( y = q(x) \), when \( x = 2 \) passes through the point \((2, 5)\) and has slope \(-9\), so has the equation:
  \[ y - 5 = -9(x - 2), \quad y = -9x + 18 + 5 = -9x + 23. \]
  Also the normal line to the curve \( y = p(x) \), when \( x = 2 \) passes through the point \((2, 5)\) and has slope \(\frac{1}{9}\), so has the equation:
  \[ y - 5 = \frac{1}{9}(x - 2), \quad 9y - 45 = x - 2, \quad x - 9y + 43 = 0. \]
Question 5

Let \( f(x) = \frac{(x-2)\ln(x)}{1+x^2} \).

- Use the definition \( f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \) to find \( f'(2) \).

We need:

\[
 f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}.
\]

Now \( f(2) = \frac{(2-2)\ln(2)}{1+2^2} = 0 \).

So the limit becomes:

\[
 f'(2) = \lim_{h \to 0} \frac{f(2+h) - 0}{h} = \lim_{h \to 0} \frac{f(2+h)}{h}
\]

\[
 = \lim_{h \to 0} \frac{(2+h-2)\ln(2+h)}{(1+(2+h)^2)} \frac{1}{h}
\]

\[
 = \lim_{h \to 0} \frac{h\ln(2+h)}{h(1+(2+h)^2)}
\]

\[
 = \lim_{h \to 0} \frac{\ln(2+h)}{1+(2+h)^2}
\]

\[
 = \frac{\ln(2+0)}{(1+(2+0)^2)} = \frac{\ln(2)}{1+2^2} = \frac{\ln(2)}{1+4} = \frac{\ln(2)}{5}.
\]

- Hence find the equation of the tangent line to the curve \( y = f(x) \) at the point with \( x = 2 \).

The point of tangency is \( (2, f(2)) = (2, 0) \) and the slope of the tangent line is \( \frac{\ln(2)}{5} \), so the required tangent line is:

\[
y - 0 = \frac{\ln(2)}{5} (x - 2), \quad 5y = \ln(2)x - 2\ln(2).
\]
Question 6

A function \( f(x) \) is given by the following formulas:

- If \( x < 2 \), then \( f(x) = 4 - (x - 1)^2 \).
- If \( x \geq 2 \), then \( f(x) = \frac{6}{x} \).

Determine with proof where the function \( f \) is continuous.
Also determine where the function \( f \) is differentiable and give formulas for its derivative.
Also plot the function \( f \) and its derivative on one graph.
Give the domain and range of the function \( f \) and the domain and range of its derivative.

Since rational functions are continuous on their domains, \( f \) is continuous everywhere, except possibly at \( x = 2 \).
Now we have:

- \( f(2) = \frac{6}{2} = 3 \).
- \( \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{6}{x} = \frac{6}{2} = 3 = f(2) \).
- \( \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} (4 - (x - 1)^2) = 4 - (2 - 1)^2 = 4 - 1^2 = 4 - 1 = 3 = f(2) \).

So \( \lim_{x \to 2} f(x) = 3 = f(2) \) and \( f \) is continuous everywhere.
Next we have for \( x < 2 \),

\[
 f'(x) = \frac{d}{dx} (4 - (x - 1)^2) = -2(x - 1)(1) = -2(x - 1) = 2 - 2x.
\]

Also we have for \( x > 2 \),

\[
 f'(x) = \frac{d}{dx} \left( \frac{6}{x} \right) = \frac{d}{dx} (6x^{-1}) = -6x^{-2} = -\frac{6}{x^2}.
\]

Then for \( x = 2 \), we have, by definition:

\[
 f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{f(x) - 3}{x - 2}.
\]
Now we have first:
\[
\lim_{x \to 2^+} \frac{f(x) - 3}{x - 2} = \lim_{x \to 2^+} \frac{\frac{6}{x} - 3}{x - 2} = \lim_{x \to 2^+} \frac{6 - 3x}{x(x - 2)} = \lim_{x \to 2^+} \frac{-3(x - 2)}{x(x - 2)} = \lim_{x \to 2^+} \frac{-3}{x} = \frac{-3}{2}.
\]

Next we have:
\[
\lim_{x \to 2^+} \frac{f(x) - 3}{x - 2} = \lim_{x \to 2^+} \frac{4 - (x - 1)^2 - 3}{x - 2} = \lim_{x \to 2^+} \frac{1 - (x^2 - 2x + 1)}{x - 2}
\]
\[
= \lim_{x \to 2^+} \frac{2x - x^2}{x - 2} = \lim_{x \to 2^+} \frac{-x(x - 2)}{x - 2} = \lim_{x \to 2^+} (-x) = -2.
\]

Since \(\lim_{x \to 2^+} \frac{f(x) - 3}{x - 2} \neq \lim_{x \to 2^+} \frac{f(x) - 3}{x - 2}\), the limit \(\lim_{x \to 2^+} \frac{f(x) - 3}{x - 2}\) does not exist and \(f\) is not differentiable at \(x = 2\).

The curve \(y = 4 - (x - 1)^2\) is a concave down parabola, symmetrical about the vertical line \(x = 1\), with vertex at \((1, 4)\), which is its highest point and which crosses the \(y\)-axis at \((0, 3)\) and the \(x\)-axis at \((-1, 0)\) and at \((3, 0)\).

Only the part of the parabola for \(x \leq 2\), so up to the point \((2, 3)\) is part of the graph of \(f\).

At \((2, 3)\) the parabola joins on to the curve \(y = \frac{6}{x}\), which is part of a rectangular hyperbola with asymptotes the axes. The hyperbola for \(x \geq 2\) is decreasing, concave up and approaches the \(x\)-axis from above as \(x \to \infty\).

The domain of \(f\) is \(\mathbb{R}\), the collection of all the real numbers.

The range of \(f\) is \((-\infty, 4]\).

For \(x < 2\), the graph of \(f'\) is a straight-line of slope \(-2\) and intercepts \((1, 0)\) and \((0, 2)\).

The end-point \((2, -2)\) is not part of the graph of \(f'\).

This portion of the graph of \(f'\) has range \((-2, \infty)\).

For \(x > 2\), the graph of \(f' = -\frac{6}{x^2}\) is a curve which is increasing and concave down and which approaches the \(x\)-axis from below as \(x \to \infty\).

The end-point \(\left(2, -\frac{3}{2}\right)\) is not part of the graph of \(f'\).

This portion of the graph of \(f'\) has range \((-\infty, \frac{3}{2}] \cup \frac{3}{2}, \infty) = \mathbb{R} - \{2\}\).

The domain of \(f'\) is \((-\infty, 2) \cup (2, \infty) = \mathbb{R} - \{2\}\).

The range of \(f'\) is \((-2, \infty)\).
Question 7

A bacterial culture has 2000 bacteria initially. After one hour there are 10000 bacteria in the culture.

- How many bacteria are there after four hours?

The population increase by a factor of \( \frac{10000}{2000} = 5 \) per hour, so by a factor of \( 5^4 = 625 \) after four hours, so after four hours the population is \( 625(2000) = 1,250,000 \), or one and one quarter million.

- Write a formula for the amount of bacteria present after \( t \) hours.

The population \( y \) of bacteria and time \( t \) hours grows by a factor of \( 5^t \), so is:

\[
y = 2000(5^t).
\]

- How long does it take for the population of bacteria to double?

The doubling time is the time for the population to increase by a factor of 2, so is given by the equation:

\[
5^t = 2,
\]

\[
t \ln(5) = \ln(2),
\]

\[
t = \frac{\ln(2)}{\ln(5)} = 0.4306765582.
\]

So the doubling time is 0.4306765582 hours, or twenty-five minutes and 50.44 seconds.

- Determine the initial rate of change with respect to time of the population of bacteria.

We have:

\[
y' = 2000(5^t) \ln(5).
\]

Putting \( t = 0 \), the initial rate of change with respect to time of the population of bacteria is

\[
2000(5^0) \ln(5) = 2000 \ln(5) = 3218.875824
\]

bacteria per hour, or 0.8941321734 bacteria per second.
• When there are ten million bacteria, half the culture is used in an experiment. When will this be?

We need;

\[ 10^7 = 2000(5^t), \]

\[ \frac{10^7}{2000} = 5^t, \]

\[ 5000 = 5^t, \]

\[ \ln(5000) = t \ln(5), \]

\[ t = \frac{\ln(5000)}{\ln(5)} = 5.292029675. \]

So there will be ten million bacteria after 5.292029675 hours, or after 5 hours, seventeen minutes and 31.31 seconds.

• The remaining half of the culture is cultured further until there are twenty million bacteria, when it is all used in an experiment. When will this be?

The remaining five million needs two doubling times to get up to twenty million, so an additional time of 0.8613531164 hours, or 51 minutes and 40.87 seconds.
**Question 8**

The vertical height $y$ meters of a particle above the ground at time $t \geq 0$ seconds is given by the formula:

$$y = 50 + 5t - 3t^2.$$ 

- Find the velocity and acceleration of the particle.

  The velocity $v$ meters per second is given by:

  $$v = y' = 5 - 6t.$$ 

  The acceleration $a$ meters per second per second is given by:

  $$a = v'' = y'' = -6.$$ 

- Find the velocity and acceleration of the particle at the time that it hits the ground.

  The particle hits the ground when $y = 0$, so when $0 = 50 + 5t - 3t^2 = (t - 5)(-3t - 10)$, so when $t = 5$ or $t = -\frac{10}{3} = -3.\overline{3}$. 

  Since we need $t > 0$, the particle hits the ground after 5 seconds. 

  The velocity is then $5 - 6(5) = -25$, or twenty-five meters per second downwards. 

  The acceleration is $-6$ meters per second per second.

- Sketch the height of the particle as a function of time and find the time that it reaches its highest point and its height and velocity at that time.

  The sketch is a standard concave down parabola, which crosses the $y$-axis at $(0, 50)$ and the $t$-axis at $\left( -\frac{10}{3}, 0 \right)$ and at $(5, 0)$, although only the part of the graph in the first quadrant is physically relevant (so for $0 \leq t \leq 5$).
The vertex of the parabola has $t$-value midway between $-\frac{10}{3}$ and 5, so:

$$t = \frac{1}{2} \left( 5 + \left( -\frac{10}{3} \right) \right) = \frac{1}{2} \left( 5 - \frac{10}{3} \right) = \frac{1}{6} (15 - 10) = \frac{5}{6}.$$

When $t = \frac{5}{6}$, the velocity is $5 - 6 \left( \frac{5}{6} \right) = 5 - 5 = 0$, as expected.

Also the height, which gives the highest point of the trajectory is then:

$$y = 50 + 5 \left( \frac{5}{6} \right) - 3 \left( \frac{5}{6} \right)^2$$

$$= 50 + \frac{25}{6} - 3 \left( \frac{25}{36} \right)$$

$$= 50 + \frac{25}{6} - \frac{25}{12}$$

$$= \frac{1}{12} (600 + 50 - 25)$$

$$= \frac{625}{12} = 52.083\ldots.$$

So the maximum height of the particle is $52.08\ldots$ meters above ground level.

Alternatively we can complete the square to find the highest point:

$$y = 50 + 5t - 3t^2 = -3 \left( t^2 - \frac{5}{3}t \right) + 50$$

$$= -3 \left( t - \frac{5}{6} \right)^2 - \frac{25}{36} + 50$$

$$= -3 \left( t - \frac{5}{6} \right)^2 + \frac{25}{12} + 50$$

$$= -3 \left( t - \frac{5}{6} \right)^2 + \frac{625}{12}.$$

This shows that the parabola is symmetrical about the vertical line $t = \frac{5}{6}$ and that the maximum height is $\frac{625}{12} = 52.08\ldots$ meters above ground, as before.