Matrix Theory and Differential Equations
Practice For the Final in BE1022
Thursday 14th December 2006 at 8.00am

Question 1

A Mars lander is approaching the moon at a speed of five kilometers per second. It decelerates smoothly at a constant rate of 50 meters per second per second.

- Write and solve the differential equation governing its motion.

If $x$ denotes its height of the lander above the surface of Mars, we have $x'' = 50$ meters per second per second.
Integrating, we get $x' = -50t + A$ and $x = -25t^2 + At + B$.

- If the lander is to land on Mars with zero speed at what height above Mars must the deceleration begin?

Let the lander hit Mars at time zero, with zero velocity.
Then $0 = 50(0) + A$, so $A = 0$ and $0 = 25(0^2) + A(0) + B$, so $B = 0$ also.
Then $x = 25t^2$.
Then $v = x' = 50t$.
When the deceleration begins, we have $v = -5000$ meters per second, so $50t = -5000$ and $t = -100$.
So the deceleration begins 100 seconds before the landing.
Then $x = 25(-100)^2 = 250,000$ meters, or 250 kilometers.
Question 2

Find general solutions of the following differential equations, explaining your reasoning: also discuss the large time behavior of your solutions, both to the future and the past.

- \[ \frac{d^2 y}{dt^2} + 25y = 250e^{-10t}. \]

  - The associated homogeneous equation is \( y'' + 25y = 0. \)

    Putting \( y = e^{mt} \) we get \( y' = me^{mt} \) and \( y'' = m^2 e^{mt}. \)

    Then we need \( 0 = y'' + 25y = e^{mt}(m^2 + 25). \)

    So \( m^2 = -25, \) \( m = \pm 5i \) and the general solution of the homogeneous system is \( y = Ae^{5it} + Be^{-5it}, \) or \( y = C \cos(5t) + D \sin(5t), \) since by Euler’s formula, we have the relations: \( e^{5it} = \cos(5t) + i \sin(5t) \) and \( e^{-5it} = \cos(5t) - i \sin(5t). \)

  - We see that the system is non-resonant, so for a particular solution we try:

    \[
    \begin{align*}
    y &= Pe^{-10t}, \\
    y' &= -10Pe^{-10t}, \\
    y'' &= 100Pe^{-10t},
    \end{align*}
    \]

    \[
    \begin{align*}
    250e^{-10t} = y'' + 25y &= 100Pe^{-10t} + 25Pe^{-10t} = 125Pe^{-10t}, \\
    \text{So } 125P &= 250, \text{ so } P = 2 \text{ and the particular solution is } y = 2e^{-10t}.
    \end{align*}
    \]

So the required general solution of the differential equation is:

\[
    y = 2e^{-10t} + C \cos(5t) + D \sin(5t).
\]

For large negative \( t, \) the exponential term dominates and \( y \) goes to infinity as \( t \to -\infty. \)

As \( t \) increases, the exponential term is modulated by the oscillatory terms and then for large positive \( t, \) the exponential term dies away, leaving an almost pure oscillation of amplitude \( \sqrt{C^2 + D^2} \) and period \( \frac{2\pi}{5}. \)
\[
\frac{dy}{dt} = y^3 + y^{-1}.
\]

This is separable:
\[
dt = \frac{dy}{y^3 + y^{-1}} = \frac{ydy}{y^4 + 1},
\]
\[
t - C = \int \frac{ydy}{y^4 + 1} = \frac{1}{2} \frac{du}{u^2 + 1} = \frac{1}{2} \arctan(u) = \frac{1}{2} \arctan(y^2).
\]

Here we put \(u = y^2\) and \(du = 2ydy\).

So we get:
\[
2(t - C) = \arctan(y^2),
\]
\[
y^2 = \tan(2(t - C)),
\]
\[
y = \pm \sqrt{\tan(2(t - C))}.
\]

The solution is defined for \(t \geq C\).

Plotting both branches of the solution, we see a bell-shaped curve, on its side (rotated through ninety degrees counter-clockwise), symmetrical about the \(x\)-axis, going to \(\pm \infty\) as \(t \to C + \frac{\pi}{4}\).

The slope is infinite at the top of the bell, at the point \((C, 0)\).

The upper branch, \(y = \sqrt{\tan(2(t - C))}\), is always increasing and is concave down for \(C \leq t \leq C + \frac{\pi}{4}\) and is otherwise concave up.

The lower branch, \(y = -\sqrt{\tan(2(t - C))}\), is always decreasing and is concave up for \(C \leq t \leq C + \frac{\pi}{4}\) and is otherwise concave down.

The two branches meet at the point \((C, 0)\).
Question 3

Consider the logistic equation with a constant harvesting rate $h$:

$$\frac{dx}{dt} = x(12 - x) - h.$$ 

Find the equilibria (if any) in the cases: $h = 0$, $h = 32$, $h = 36$ and $h = 40$. Also discuss the stability of the equilibria. In each case, if the initial population is $x = 10$, discuss what happens to the population as time passes.

The equilibria are the roots of $x(12 - x) - h$.

- When $h = 0$, the roots of $x(12 - x) = 0$ are $x = 0$ and $x = 12$.
  When $x > 12$, $x' < 0$; when $0 < x < 12$, $x' > 0$, so $x = 12$ is stable.
  When $x < 0$, $x' < 0$; when $0 < x < 12$, $x' > 0$, so $x = 0$ is unstable.
  If $x$ is initially 10, then the solution, for positive $t$, is increasing and concave down and rises towards $x = 12$, which it approaches from below as $t \to \infty$.

- When $h = 32$, then if $x(12 - x) - 32 = 0$, we get $x^2 - 12x + 32 = 0$, or $(x - 4)(x - 8) = 0$, so the equilibria are $x = 4$ and $x = 8$.
  When $x > 8$, $x' < 0$; when $4 < x < 8$, $x' > 0$, so $x = 8$ is stable.
  When $x < 4$, $x' < 0$; when $4 < x < 8$, $x' > 0$, so $x = 4$ is unstable.
  If $x$ is initially 10, then the solution, for positive $t$, is decreasing and concave up and decreases towards $x = 8$, which it approaches from above as $t \to \infty$.

- When $h = 36$, then if $x(12 - x) - 36 = 0$, we get $x^2 - 12x + 36 = 0$, or $(x - 6)^2 = 0$, so the only equilibrium is $x = 6$.
  When $x > 6$, $x' < 0$; when $x < 6$, $x' < 0$, so $x = 6$ is semi-stable: stable from above and unstable from below.
  If $x$ is initially 10, then the solution, for positive $t$, is decreasing and concave up and decreases towards $x = 6$, which it approaches from above as $t \to \infty$.

- When $h = 40$, then we have $x' = x(12 - x) - 40 = -(x^2 - 12x + 40) = -(x - 6)^2 - 4 \leq -4$, so there are no equilibria: $x$ decreases steadily and the population is reduced to zero in finite time.
  If $x$ is initially 10, then the solution decreases to zero no later than $t = 2.5$.
  In fact the exact solution is $x = \frac{10 \cos(2t) + 10 \sin(2t)}{\cos(2t) + 2 \sin(2t)}$, which goes to zero when $t = \frac{3 \pi}{8} = 1.178097245$.
  Then the solution goes to minus infinity at $t = \frac{1}{2}(\pi - \arctan(\frac{1}{2}) = 1.338972522$. 

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Question 4

Find a basis for the row space and a basis for the column space of the following matrix:

\[
A = \begin{bmatrix}
1 & 6 & 2 & -1 \\
2 & 8 & 3 & -4 \\
2 & 5 & 5 & -5 \\
3 & 3 & 5 & -10
\end{bmatrix}.
\]

Also find a basis for the solution space of the linear system \(AX = 0\), where \(X\) is a column vector with four components.

We row reduce the matrix \(A\):

\[
\begin{bmatrix}
1 & 6 & 2 & -1 \\
2 & 8 & 3 & -4 \\
2 & 5 & 5 & -5 \\
3 & 3 & 5 & -10
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 6 & 2 & -1 \\
0 & -4 & -1 & -2 \\
0 & -7 & 1 & -3 \\
0 & -15 & -1 & -7
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 6 & 2 & -1 \\
0 & -4 & -1 & -2 \\
0 & 1 & 3 & 1 \\
0 & 1 & 3 & 1
\end{bmatrix}
\]

Here we did the row operations: \(R_2 \rightarrow -2R_1 + R_2\), \(R_3 \rightarrow -2R_1 + R_3\), \(R_4 \rightarrow -3R_1 + R_4\), \(R_4 \rightarrow -4R_2 + R_4\), \(R_3 \rightarrow -2R_2 + R_3\), \(R_4 \rightarrow -3R_2 + R_4\), \(R_3 \rightarrow \frac{1}{11}R_3\), \(R_2 \rightarrow -3R_3 + R_2\), \(R_1 \rightarrow 16R_3 + R_1\).

The matrix \(A\) is rank 3.

A basis for the row space is the (ordered) set of vectors \([1, 0, 0, -\frac{45}{11}], [0, 1, 0, \frac{5}{11}], [0, 0, 1, \frac{2}{11}]\).

A basis for the column space is assembled from the pivot columns, so here the first three columns, corresponding to the dependent variables, for the solution of the homogeneous system, so is the ordered set \([1, 2, 2, 3], [6, 8, 5, 3], [2, 3, 5, 5]\) (written as columns).

Writing \(X = [x, y, z, t]^T\), transposed the reduced homogenous system \(AX = 0\) is:

\[
x - \frac{45}{11}t = 0, \quad y + \frac{5}{11}t = 0, \quad z + \frac{2}{11}t = 0.
\]

We pick \(t = 11s\), with \(s\) arbitrary, giving \(x = 45s\), \(y = -5s\) and \(z = -2s\), so the general solution of the homogeneous system \(AX = 0\) is \(X = s[45, -5, -2, 11]^T\). So a basis for the solution space is the vector \([45, -5, -2, 11]^T\), written as a column vector.
Question 5

Let \( A = [3, 2, -1], B = [6, -2, -1], C = [10, 1, -6] \) and \( D = [8, 9, 3] \).

- Find the equation of the plane through \( A, B \) and \( C \).

We use the determinant method: the required equation is:

\[
0 = \det \begin{vmatrix} 1 & x & y & z \\ 1 & 3 & 2 & -1 \\ 1 & 6 & -2 & -1 \\ 1 & 10 & 1 & -6 \end{vmatrix} = \det \begin{vmatrix} 0 & x-3 & y-2 & z+1 \\ 1 & 3 & 2 & -1 \\ 0 & 3 & -4 & 0 \\ 0 & 7 & -1 & -5 \end{vmatrix}
\]

\[
= - \det \begin{vmatrix} x-3 & y-2 & z+1 \\ 3 & -4 & 0 \\ 7 & -1 & -5 \end{vmatrix}
\]

\[
= -(z+1)(-3+28) + 5(-4(x-3) - 3(y-2))
\]

\[
= -25z - 25 - 20x + 60 - 15y + 30
\]

\[
= -20x - 15y - 25z + 65.
\]

Here we first did the row operations \( R_1 \rightarrow -R_2 + R_1, R_3 \rightarrow -R_2 + R_3 \) and \( R_4 \rightarrow -R_2 + R_4 \).

Then we Laplace expanded down the first column.

Then we Laplace expanded down the third column. After dividing by 5, the equation of the plane becomes \( 4x + 3y + 5z = 13 \).

It is easily checked that each of the points, \( A, B \) and \( C \), satisfies this equation, as required.
Find the angles of the triangle $ABC$.

We have:

- $BC = C - B = [4, 3, -5]$, $a = |BC| = \sqrt{16 + 9 + 25} = \sqrt{50} = 5\sqrt{2},$
- $CA = A - C = [-7, 1, 5]$, $b = |CA| = \sqrt{49 + 1 + 25} = \sqrt{75} = 5\sqrt{3},$
- $AB = B - A = [3, -4, 0]$, $c = |AB| = \sqrt{9 + 16 + 0} = \sqrt{25} = 5$.

We see that $b^2 = 75 = a^2 + c^2 = 50 + 25$, so the triangle is right-angled, with hypotenuse $b$.

So the angle at $B$ is $\frac{\pi}{2}$ radians, or ninety degrees and the angle at $A$ is

\[
\arctan\left(\frac{a}{c}\right) = \arctan(\sqrt{2}) = 0.9553166180\ \text{radians}, \quad \text{or} \quad 54.73561028\ \text{degrees}.
\]

Finally the angle at $C$ is $\arctan\left(\frac{c}{a}\right) = \arctan\left(\frac{1}{\sqrt{2}}\right) = 0.6154797085\ \text{radians}, \quad \text{or} \quad 35.26438965\ \text{degrees}.$

Find the area $\Delta$ of the triangle $ABC$.

We have $\Delta = \frac{1}{2} ac = \frac{25}{2}\sqrt{2}$.

Find the volume $V$ of the tetrahedron $ABCD$.

If the equation of the plane is written in the form $\overrightarrow{N} \cdot \overrightarrow{X} - c = 0$, where $\overrightarrow{N} = [4, 3, 5]$ is the normal, $c = 13$ and $\overrightarrow{X} = [x, y, z]$, then the height $h$ of $D$ above the plane $ABC$ is:

\[
h = \frac{1}{|\overrightarrow{N}|} |\overrightarrow{N} \cdot \overrightarrow{D} - c| = \frac{1}{\sqrt{4^2 + 3^2 + 5^2}} |[4, 3, 5], [8, 9, 3] - 13|
\]

\[
= \frac{1}{\sqrt{16 + 9 + 25}} |32 + 27 + 15 - 13| = \frac{61}{5\sqrt{2}}.
\]

Then the volume $V$ of the tetrahedron is:

\[
V = \frac{1}{3} \Delta h = \frac{1}{3} \frac{25}{2} \sqrt{2} \frac{1}{5\sqrt{2}} 61 = \frac{305}{6} = 50.83.
\]
Question 6

A mass-spring-dashpot system has mass \( m = 15 \) kilos, spring constant 1 Newton per meter and dashpot constant 8 Newton seconds per meter and is subject to a driving force of \( 20 \cos(t) + 30 \sin(t) \) Newtons.

Find the solution of the system starting at time zero, from rest, at displacement 6 meters from equilibrium and discuss its behavior as a function of time.

The differential equation governing the motion is:

\[
15x'' + 8x' + x = 20 \cos(t) + 30 \sin(t).
\]

The associated homogeneous system is:

\[
0 = (15D^2 + 8D + 1)x = (5D + 1)(3D + 1)x.
\]

The roots are \(-\frac{1}{5}\) and \(-\frac{1}{3}\), so the general solution of the homogeneous system is:

\[
x = Ae^{-\frac{t}{5}} + Be^{-\frac{t}{3}}.
\]

For a particular solution, since the system is non-resonant, we put:

\[
x = P \cos(t) + Q \sin(t).
\]

Then we have:

\[
x' = Q \cos(t) - P \sin(t),
x'' = -P \cos(t) - Q \sin(t),
20 \cos(t) + 30 \sin(t) = 15x'' + 8x' + x = \cos(t)(-15P + 8Q + P) + \sin(t)(-15Q - 8P + Q),
20 = -14P + 8Q,
30 = -8P - 14Q,
10 = -7P + 4Q, \quad 15 = -4P - 7Q,
40 = -28P + 16Q, \quad 105 = -28P - 49Q,
-65 = 65Q, \quad Q = -1, \quad 8P = -14Q - 30 = 14 - 30 = -16,
P = -2, \quad Q = -1.
\]

So the general solution is:

\[
x = Ae^{-\frac{t}{5}} + Be^{-\frac{t}{3}} - 2 \cos(t) - \sin(t).
\]
Then we have:

\[ x' = -\frac{A}{5}e^{-\frac{t}{4}} - \frac{B}{3}e^{-\frac{t}{3}} + 2\sin(t) - \cos(t). \]

The initial conditions are then:

\[ 6 = A + B - 2, \quad 0 = -\frac{A}{5} - \frac{B}{3} - 1, \]
\[ A + B = 8, \quad 3A + 5B = -15, \]
\[ 3A + 3B = 24, \]
\[ 2B = -39, \quad B = \frac{39}{2}, \]
\[ A = 8 - B = \frac{55}{2}. \]

So the required solution is:

\[ x = \frac{55}{2}e^{-\frac{t}{4}} - \frac{39}{2}e^{-\frac{t}{3}} - 2\cos(t) - \sin(t). \]

For large negative \( t \), the solution blows up going to minus infinity, as \( t \to -\infty \), with the term \(-\frac{39}{2}e^{-\frac{t}{3}}\) dominating.

For \( t \) larger than about 20, the exponential terms die away, leaving an oscillation of period \( 2\pi \) and amplitude \( \sqrt{5} \).

The solution has an absolute maximum at \( t = 3.296921966 \) when \( x = 9.855141617 \), but no absolute minimum.
Question 7

Solve the differential system:

\[ X' = AX + F, \quad A = \begin{bmatrix} -3 & -1 \\ 2 & -6 \end{bmatrix}, \quad F = \begin{bmatrix} 6e^{-t} \\ 12e^{-t} \end{bmatrix} \]

with the initial condition \( X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).

Discuss the behavior of the system as time varies.

The eigen-value equation for \( A \) is:

\[
0 = \det(A - \lambda I) = \det \begin{vmatrix} -3 - \lambda & -1 \\ 2 & -6 - \lambda \end{vmatrix} = (-3 - \lambda)(-6 - \lambda) + 2 = \lambda^2 + 9\lambda + 20 = (\lambda + 4)(\lambda + 5).
\]

So the eigen-values are \(-4\) and \(-5\).

- When \( \lambda = -4 \), the eigen-vector equation is:
  \[
  0 = \det \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p - q \\ 2p - 2q \end{bmatrix}.
  \]

So \( p = q \) and we may take the eigen-vector to be \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

- When \( \lambda = -5 \), the eigen-vector equation is:
  \[
  0 = \det \begin{vmatrix} 2 & -1 \\ 2 & -1 \end{vmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 2p - q \\ 2p - q \end{bmatrix}.
  \]

So \( q = 2p \) and we may take the eigen-vector to be \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \).

So an invertible matrix of eigen-vectors is:

\[
P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.
\]

Note that \( \det(P) = 1 \) and \( P^{-1} \) is the matrix:

\[
P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.
\]
The corresponding fundamental solution of the homogeneous matrix system $X' = AX$ is:

$$\Phi(t) = \begin{bmatrix} e^{-4t} & e^{-5t} \\ e^{-4t} & 2e^{-5t} \end{bmatrix}, \quad \Phi(0) = P.$$

We see that the forcing term is non-resonant, so for a particular solution, we put $X = e^{-t} \begin{bmatrix} c \\ d \end{bmatrix}$.

Substituting, we need:

$$F = \begin{bmatrix} 6e^{-t} \\ 12e^{-t} \end{bmatrix} = X' - AX$$

$$= -e^{-t} \begin{bmatrix} c \\ d \end{bmatrix} - \begin{bmatrix} -3 & -1 \\ 2 & -6 \end{bmatrix} e^{-t} \begin{bmatrix} c \\ d \end{bmatrix},$$

$$= e^{-t} \begin{bmatrix} -c \\ -d \end{bmatrix} + e^{-t} \begin{bmatrix} 3c + d \\ -2c + 6d \end{bmatrix},$$

$$\begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 2c + d \\ -2c + 5d \end{bmatrix}.$$

$6 = 2c + d$, $12 = -2c + 5d$, $18 = 6 + 12 = (2c + d) + (-2c + 5d) = 6d$,

$$d = 3, \quad 2c + 3 = 6, \quad c = \frac{3}{2}.$$

So a particular solution is:

$$X = \frac{3}{2} e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

So the general solution of the given system is:

$$X = \Phi(t)C + \frac{3}{2} e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The initial condition is:

$$0 = \Phi(0)C + \frac{3}{2} e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = PC + \frac{3}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$C = -\frac{3}{2} P^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\frac{3}{2} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\frac{3}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$
So the required solution is:

\[
X = -\frac{3}{2} \begin{pmatrix} e^{-4t} & e^{-5t} \\ e^{-4t} & 2e^{-5t} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{3}{2} e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
= \frac{3}{2} (e^{-t} - e^{-5t}) \begin{pmatrix} 1 \\ 2 \end{pmatrix}.
\]

So, writing \( X = \begin{pmatrix} x \\ y \end{pmatrix} \), the motion is entirely along the line \( y = 2x \).

As \( t \to -\infty \), the motion starts at infinity in the fourth quadrant along the line.
As \( t \) increases, the motion moves towards the origin, reaching it at \( t = 0 \).
Then the motion moves into the first quadrant, reaching its furthest point from the origin in that quadrant, when \( t = \frac{1}{4} \ln(5) \), when:

\[
X = 6(5^{-\frac{3}{4}}) \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.802488366 \\ 1.604976732 \end{pmatrix}.
\]

Then as \( t \to \infty \), the solution goes back towards its limiting position, the origin:
\( X \to 0 \).
Question 8

Write the differential equation \( y'' - 8y' + 12y = 0 \) as a first order matrix system: 
\[ X' = AX, \] 
where \( X \) is the column vector \( X = \begin{bmatrix} y \\ y' \end{bmatrix} \).

Find the eigen-values and eigenvectors of the matrix \( A \) and hence find the general solution of the matrix system.

Find the solution with initial condition \( X(0) = \begin{bmatrix} 4 \\ -2 \end{bmatrix} \) and discuss its behavior as a function of time.

We have:
\[ X' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ 8y' - 12y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & 8 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = AX, \]
\[ A = \begin{bmatrix} 0 & 1 \\ -12 & 8 \end{bmatrix}. \]

The eigen-value equation for \( A \) is:
\[ 0 = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -12 & 8 - \lambda \end{bmatrix} = -\lambda(8 - \lambda) + 12 = \lambda^2 - 8\lambda + 12 = (\lambda - 2)(\lambda - 6). \]

- When \( \lambda = 2 \), the eigen-vector equation is:
  \[ \det \begin{bmatrix} -2 & 1 \\ -12 & 6 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} -2p + q \\ -12p + 6q \end{bmatrix}. \]
  So \( q = 2p \) and we may take the eigen-vector to be \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \).

- When \( \lambda = 6 \), the eigen-vector equation is:
  \[ \det \begin{bmatrix} -6 & 1 \\ -12 & 2 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} -6p + q \\ -12p + 2q \end{bmatrix}. \]
  So \( q = 6p \) and we may take the eigen-vector to be \( \begin{bmatrix} 1 \\ 6 \end{bmatrix} \).
Then the general solution of the matrix system is:

\[ X = Ae^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + Be^{6t} \begin{pmatrix} 1 \\ 6 \end{pmatrix}. \]

Putting \( t = 0 \), we need:

\[
X(0) = \begin{pmatrix} 4 \\ -2 \end{pmatrix} = A \begin{pmatrix} 1 \\ 2 \end{pmatrix} + B \begin{pmatrix} 1 \\ 6 \end{pmatrix},
\]

\[
A + B = 4, \quad 2A + 6B = -2,
\]

\[
A + 3B = -1, \quad 2B = -5,
\]

\[
B = \frac{-5}{2}, \quad A = 4 - B = \frac{13}{2}.
\]

So the required solution is:

\[
X = \frac{1}{2} \begin{pmatrix} 13e^{2t} - 5e^{6t} \\ 26e^{2t} - 30e^{6t} \end{pmatrix}.
\]

Writing \( X = \begin{pmatrix} x \\ y \end{pmatrix} \), the \((x, y)\) plot is a hairpin shape, going in towards the origin in the first quadrant as \( t \to -\infty \) in the direction with slope 2, since the \( e^{2t} \) term dominates.

Then the curve is concave down and increasing until \( t = \frac{1}{4}(\ln(13) - \ln(5) - 2\ln(3)) = -0.3104282830 \), at the point \((x, y) = \frac{26\sqrt{65}}{135}(2, 3) = (3.105462243, 4.658193364)\).

Then it decreases and the slope goes to infinity as it crosses the \( x \)-axis into the fourth quadrant at \( x = \frac{12}{25}\sqrt{195} = 4.034113789 \).

Below that point the curve is concave up and increasing (as \( x \) increases).

As \( t \) goes to infinity, the graph goes to infinity in the direction of the line \( y = 6x \), in the third quadrant, which it approaches from above. The curve crosses the \( y \)-axis into the third quadrant when \( t = \frac{1}{4}(\ln(13) - \ln(5)) \), when \( y = -\frac{26}{5}\sqrt{65} = -41.92374027 \).
Question 9

Find a fundamental $2 \times 2$ matrix solution $\Phi$ for the system: $X' = AX$, where $A$ is the following matrix:

$$A = \begin{bmatrix} 5 & -3 \\ 8 & -6 \end{bmatrix}$$

Hence find the solution $X$ of the system with the initial condition $X(0) = \begin{bmatrix} 15 \\ -10 \end{bmatrix}$ and discuss the behavior of the solution as a function of time.

Also give a formula for the matrix exponential $e^{At}$.

The eigen-value equation for $A$ is:

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} 5 - \lambda & -3 \\ 8 & -6 - \lambda \end{bmatrix}$$

$$= (5 - \lambda)(-6 - \lambda) + 24 = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2).$$

So the eigen-values are 2 and $-3$.

- When $\lambda = 2$, the eigen-vector equation is:

$$0 = \det \begin{bmatrix} 3 & -3 \\ 8 & -8 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 3p - 3q \\ 8p - 8q \end{bmatrix}.$$  

So $p = q$ and we may take the eigen-vector to be $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

- When $\lambda = -3$, the eigen-vector equation is:

$$\det \begin{bmatrix} 8 & -3 \\ 8 & -3 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 8p - 3q \\ 8p - 3q \end{bmatrix}.$$  

So $3q = 8p$ and we may take the eigen-vector to be $\begin{bmatrix} 3 \\ 8 \end{bmatrix}$.

So an invertible matrix of eigen-vectors is:

$$P = \begin{bmatrix} 1 & 3 \\ 1 & 8 \end{bmatrix}.$$
The corresponding fundamental solution of the matrix system is:

\[
\Phi = \begin{bmatrix} e^{2t} & 3e^{-3t} \\ e^{2t} & 8e^{-3t} \end{bmatrix}, \quad \Phi(0) = P.
\]

The required solution is then \( X(t) = \Phi(t)C \).

Putting \( t = 0 \), we need:

\[
X(0) = \Phi(0)C = PC,
\]

\[
C = P^{-1}X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 15 \\ -10 \end{bmatrix} = \begin{bmatrix} 30 \\ -5 \end{bmatrix},
\]

\[
X = \Phi C = \begin{bmatrix} e^{2t} & 3e^{-3t} \\ e^{2t} & 8e^{-3t} \end{bmatrix} \begin{bmatrix} 30 \\ -5 \end{bmatrix} = \begin{bmatrix} 30e^{2t} - 15e^{-3t} \\ 30e^{2t} - 40e^{-3t} \end{bmatrix}.
\]

Writing \( X = \begin{bmatrix} x \\ y \end{bmatrix} \), the \((x,y)\)-plot is a slightly bent line, everywhere increasing and concave down.

It goes towards the asymptote \( y = x \), which it approaches from below in the first quadrant, as \( t \to \infty \).

It goes towards the asymptote \( y = \frac{8}{3}x \), which it approaches from below in the third quadrant, as \( t \to -\infty \).

It crosses the \( x \)-axis from the first quadrant into the fourth quadrant when \( t = -\frac{1}{5}(\ln(3) - \ln(4)) = 0.05753641450 \), when \( x = 25(3^{\frac{2}{3}}2^{-\frac{2}{3}}) = 21.03665898 \).

It crosses the \( y \)-axis from the third quadrant into the fourth quadrant when \( t = -\frac{1}{5}\ln(2) = -0.1386294361 \), when \( x = -25(2^{\frac{2}{3}}) = -37.89291418 \).

Finally, we have:

\[
e^{At} = \Phi(t)\Phi(0)^{-1} = \Phi(t)P^{-1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} e^{2t} & 3e^{-3t} \\ e^{2t} & 8e^{-3t} \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 8e^{2t} - 3e^{-3t} & -3e^{2t} + 3e^{-3t} \\ 8e^{2t} - 8e^{-3t} & -3e^{2t} + 8e^{-3t} \end{bmatrix}.
\]

Note that when \( t = 0 \), this reduces to the identity matrix, as it should.
Question 10

Tanks $A$ and $B$ are filled with brine.
Tank $A$ has 80 gallons of brine and tank $B$ has 20 gallons of brine.
Brine flows from $A$ into $B$ along a pipe at a rate of 5 gallons per minute.
Brine flows from $B$ into $A$ along a second pipe also at a rate of 5 gallons per minute.
The brine is always kept well-mixed.
Initially there are 12 pounds of salt in tank $A$ and 6 pounds of salt in tank $B$.
Discuss the behavior of the amount of salt in each tank as a function of time.

Note that each tank contains the same amount of fluid at all times.
Let tank $A$ have $x$ pounds of salt in it at time $t$ minutes.
Let tank $B$ have $y$ pounds of salt in it at time $t$ minutes.
Then the rate of inflow of salt into tank $A$ from tank $B$ is $5 \frac{y}{20} = \frac{y}{4}$ pounds per minute.
Also the rate of outflow of salt from tank $A$ into tank $B$ is $5 \frac{x}{80} = \frac{x}{16}$ pounds per minute.
So we have the differential system:
\[
\frac{dx}{dt} = -\frac{x}{16} + \frac{y}{4},
\frac{dy}{dt} = \frac{x}{16} - \frac{y}{4}.
\]
Note that $\frac{dx}{dt} + \frac{dy}{dt} = 0$, so $x + y$ is constant, so $x + y = 18$, since initially $x = 12$ and $y = 6$.
So $y = 18 - x$ and the equation for $\frac{dx}{dt}$ becomes:
\[
\frac{dx}{dt} = -\frac{x}{16} + \frac{18 - x}{4} = \frac{9}{2} - \frac{5x}{16},
\Rightarrow \frac{dx}{dt} + \frac{5x}{16} = \frac{9}{2}.
\]
A particular solution is $x = C$, a constant, where $\frac{5C}{16} = \frac{9}{2}$, so $C = \frac{72}{5}$.
The general solution of the associated homogeneous equation $\frac{dx}{dt} + \frac{5x}{16} = 0$ is:
\[
x = Ae^{-\frac{5}{16}t}.
\]
So the general solution for $x$ is:

$$x = Ae^{-\frac{3}{5}t} + \frac{72}{5},$$

Putting $t = 0$, since then $x = 12$, we need:

$$12 = A + \frac{72}{5},$$

$$A = -\frac{12}{5},$$

$$x = \frac{12}{5}(6 - e^{-\frac{3}{5}t}),$$

$$y = 18 - x = \frac{6}{5}(3 + 2e^{-\frac{3}{5}t}).$$

So as $t$ increases from zero, the amount of salt in tank $A$ steadily increases, rapidly approaching its asymptotic value of $\frac{72}{5} = 14.4$ pounds, whereas the amount of salt in tank $B$ steadily decreases, rapidly approaching its asymptotic value of $\frac{18}{5} = 3.6$ pounds.

Note that the asymptotic value corresponds to a uniform density of salt of 0.18 pounds of salt per gallon in the whole system.