Matrix Theory and Differential Equations
Homework 4 Solutions, 9/21/6

Question 1

The dynamics of a certain spring are governed by the differential equation \( \frac{d^2y}{dt^2} = -9y \), where \( y \) meters is the extension of the spring from its rest position and \( t \) is measured in seconds.

Solve for \( y \) as a function of time given that \( y = 4 \) m. and \( \frac{dy}{dt} = 9 \) m/s. at \( t = 0 \).

Putting \( y = e^{mt} \), we get \( \frac{dy}{dt} = me^{mt} \) and \( \frac{d^2y}{dt^2} = m^2 e^{mt} \).

Then the differential equation becomes:

\[
0 = \frac{d^2y}{dt^2} + 9y = e^{mt} (m^2 + 9), \quad m^2 + 9 = 0, \quad m^2 = -9, \quad m = \pm 3i.
\]

So we have solutions \( e^{3it} \) and \( e^{-3it} \).

Using Euler’s formula these solutions are \( \cos(3t) + i \sin(3t) \) and \( \cos(3t) - i \sin(3t) \), respectively.

By combining these appropriately, using the Principle of Linear Superposition, we get the two real solutions \( \cos(3t) \) and \( \sin(3t) \).

Then the general solution is \( y = A \cos(3t) + B \sin(3t) \).

So we get \( \frac{dy}{dt} = -3A \sin(3t) + 3B \cos(3t) \).

Putting \( t = 0 \), \( y = -4 \) and \( \frac{dy}{dt} = 9 \), since \( \sin(0) = 0 \) and \( \cos(0) = 1 \), we get: \( -4 = A \) and \( 3B = 9 \), so \( B = 3 \) and the required solution for \( y \) is: \( y = -4 \cos(3t) + 3 \sin(3t) \). The amplitude is then \( \sqrt{(-4)^2 + 3^2} = 5 \). We can also write this solution as \( y = 5 \sin(3t + C) = 5 \sin(3t) \cos(C) + 5 \cos(3t) \sin(C) \), so \( 5 \cos(C) = -4 \) and \( 5 \sin(C) = 3 \), so \( \tan(C) = -\frac{3}{4} \), with \( C \) in the second quadrant. So \( C = \pi - \arctan \left( \frac{3}{4} \right) \). The motion is simple harmonic of amplitude 5 and period \( \frac{2\pi}{3} \). The graph is a standard sine curve, crossing the axis at \( t = \frac{n\pi - C}{3} \), for \( n \) any integer. The maximum extension of the spring is 5 meters.
Question 2

Vertical motion under the earth’s gravity is governed by the differential equation:

\[
\frac{d^2r}{dt^2} = -\frac{GM}{r^2}.
\]

Show that if \( v = \frac{dr}{dt} \), then \( v^2 - \frac{2GM}{r} \) is constant.

Suppose that at radius \( R \) a particle has vertical velocity \( V \) and suppose that the particle just escapes to infinity (i.e. as \( r \to \infty \), we have \( v \to 0^+ \)).

Show that \( V^2 = \frac{2GM}{R} \).

\( V \) is then called the escape velocity.

For a black hole, the escape velocity from the horizon at radius \( R \) is the speed of light \( c \), so we then have: \( R = \frac{2GM}{c^2} \).

Compute \( R \) for the earth (use \( M = 6(10^{24}) \) kilos, \( c = 3(10^8) \) meters per second and \( G = \frac{2}{3}(10^{-10}) \) in metric units).

Using \( r \) and \( v \) as the basic variables, we have, by the chain rule:

\[
\frac{d^2r}{dt^2} = \frac{d}{dt} \left( \frac{dr}{dt} \right) = \frac{dv}{dt} = \frac{dr}{dt} \frac{dv}{dr} = v \frac{dv}{dr}.
\]

So the differential equation \( \frac{d^2r}{dt^2} = -\frac{GM}{r^2} \) becomes:

\[
v \frac{dv}{dr} = -\frac{GM}{r^2},
\]

\[
vdv = -GMr^{-2}dr,
\]

\[
\int 2vdv = -\int 2GMr^{-2}dr,
\]

\[
v^2 = 2GMr^{-1} + C,
\]

\[
v^2 - \frac{2GM}{r} = C.
\]

So \( v^2 - \frac{2GM}{r} \) is constant, as required.
If $r$ goes to infinity as $v$ goes to zero, we have:

$$C = \lim_{r \to \infty} \left( v^2 - \frac{2GM}{r} \right) = 0^2 - 0 = 0.$$ 

Since $C$ is zero, we now have the required relation:

$$v^2 = \frac{2GM}{r}.$$ 

Then at radius $R$ and velocity $V$, we have the relations:

$$V^2 = \frac{2GM}{R},$$

$$R = \frac{2GM}{V^2}.$$ 

When $V = c = 3(10^8)$, $M = 6(10^{24})$ and $G = \frac{2}{3}(10^{-10})$, we get:

$$R = \frac{2GM}{c^2} = \frac{2 \left( \frac{2}{3} \right) (10^{-10})(6)(10^{24})}{9(10^{16})} = \frac{8}{900}.$$ 

So $R = \frac{8}{9}$ centimeters.

The value of this radius is called the Schwarzschild radius for the earth. If all the matter in the earth were squashed into a black hole this radius is effectively the "size" of the black hole. Physically, it is the radius inside of which no light can escape to infinity. The surface of constant radius the Schwarzschild radius is called the horizon. Anything inside the horizon is doomed to crash into the singularity at the center of the black hole!
Question 3

An arrow shot vertically upwards experiences the deceleration of gravity of size $32$ feet per second per second and a deceleration due to air resistance of size $\frac{v^2}{200}$. If the arrow has initial velocity $200$ feet per second, find how high the arrow goes.

If the upward acceleration of the arrow is $a$ feet per second per second and the velocity is $v$ feet per second, we have:

$$a = -32 - \frac{v^2}{200}.$$  

But $a = \frac{dv}{dt}$, where $t$ is measured in seconds, so we have:

$$\frac{dv}{dt} = -32 - \frac{v^2}{200},$$  

$$-32 - \frac{v^2}{200} = dt,$$  

$$\frac{200dv}{6400 + v^2} = -dt,$$  

$$-t + C = 200 \int \frac{dv}{80^2 + v^2} = \frac{200}{80} \arctan \left( \frac{v}{80} \right) = \frac{5}{2} \arctan \left( \frac{v}{80} \right).$$

Put $t = 0$ and $v = 200$, giving

$$C = \frac{5}{2} \arctan \left( \frac{200}{80} \right) = \frac{5}{2} \arctan \left( \frac{5}{2} \right),$$  

$$-t + \frac{5}{2} \arctan \left( \frac{5}{2} \right) = \frac{5}{2} \arctan \left( \frac{v}{80} \right),$$  

$$-\frac{2t}{5} + \arctan \left( \frac{5}{2} \right) = \arctan \left( \frac{v}{80} \right),$$  

$$\frac{v}{80} = \tan \left( -\frac{2t}{5} + \arctan \left( \frac{5}{2} \right) \right),$$  

$$v = 80 \tan \left( -\frac{2t}{5} + \arctan \left( \frac{5}{2} \right) \right).$$
Next, since $v = \frac{dx}{dt}$, we integrate $v$ to get $x$:

$$\frac{dx}{dt} = 80 \tan \left( -\frac{2t}{5} + \arctan \left( \frac{5}{2} \right) \right),$$

$$x = \int 80 \tan \left( -\frac{2t}{5} + \arctan \left( \frac{5}{2} \right) \right) dt$$

$$= 200 \ln \left( A \cos \left( -\frac{2t}{5} + \arctan \left( \frac{5}{2} \right) \right) \right).$$

Putting $t = 0$ and $x = 0$, we get:

$$0 = \ln(A \cos \left( \arctan \left( \frac{5}{2} \right) \right)) = \ln(1), \quad A = \frac{1}{\cos \left( \arctan \left( \frac{5}{2} \right) \right)}.$$

So we get, finally:

$$x = 200 \ln \left( \frac{\cos \left( -\frac{2t}{5} + \arctan \left( \frac{5}{2} \right) \right)}{\cos \left( \arctan \left( \frac{5}{2} \right) \right)} \right).$$

The formula $v = \tan \left( -\frac{2t}{5} + \arctan \left( \frac{5}{2} \right) \right)$, shows that the first time that for which $v$ becomes zero occurs when:

$$0 = -\frac{2t}{5} + \arctan \left( \frac{5}{2} \right), \quad t = \frac{5}{2} \arctan \left( \frac{5}{2} \right) = 2.9757.$$

For this value of $t$, the corresponding value of $x$ is:

$$x = 200 \ln \left( \cos \left( \arctan \left( \frac{5}{2} \right) \right) \right) = -200 \ln \left( \cos \left( \arctan \left( \frac{5}{2} \right) \right) \right).$$

Now the angle $\theta = \arctan \left( \frac{5}{2} \right)$ is the angle of a right-angled triangle, with opposite side 5 and adjacent side 2.

Then the hypotenuse is $\sqrt{2^2 + 5^2} = \sqrt{29}$.

So $\cos(\theta) = \frac{2}{\sqrt{29}}$.

So the arrow goes as high as:

$$x = -200 \ln \left( \cos \left( \arctan \left( \frac{5}{2} \right) \right) \right) = -200 \ln \left( \frac{2}{\sqrt{29}} \right)$$

$$= 100 \ln(29) - 200 \ln(2) = 198.1.$$

So the arrow rises to a maximum height of 198.1 feet in 2.9757 seconds.
A much quicker method is to go directly for the relation between the velocity and position.

We use the standard formula: \( a = v \frac{dv}{dx} \), which we proved in class, so we have the differential equation:

\[
\frac{dv}{dx} = -32 - \frac{v^2}{200},
\]

\[-200v \frac{dv}{dx} = 6400 + v^2,\]

\[\frac{-200vdv}{6400 + v^2} = dx,\]

\[x + C = -100 \int \frac{2vdv}{6400 + v^2} = -100 \ln(6400 + v^2).\]

Initially we have \( x = 0 \) and \( v = 200 \), so we get:

\[C = -100 \ln(6400 + 200^2) = -100 \ln(46400),\]

\[x - 100 \ln(46400) = -100 \ln(6400 + v^2),\]

\[x = 100 \ln(46400) - 100 \ln(6400 + v^2) = 100 \ln \left( \frac{46400}{6400 + v^2} \right).\]

Now we put \( v = 0 \), to find the highest point of the trajectory, giving:

\[x = 100 \ln \left( \frac{46400}{6400} \right) = 100 \ln \frac{29}{4} = 198.1.\]

So the arrow rises to a maximum height of 198.1 feet as before. Note that this way, without more work we do not know when it gets to the top.
Question 4

Plot the slope field for the equation:
\[
\frac{dx}{dt} = x(9 - x^2).
\]

Discuss the stability of the critical points of the differential equation.

The critical points are the zeros of the function:
\[
x(9 - x^2) = x(3 - x)(3 + x),
\]
so the critical points are at \(x = 0\), \(x = 3\) and \(x = -3\).

- For \(x > 3\) the slopes are negative, so the solutions decrease towards \(x = 3\).
- For \(0 < x < 3\) the slopes are positive, so the solutions increase towards \(x = 3\) and away from \(x = 0\).
- For \(-3 < x < 0\) the slopes are negative, so the solutions decrease towards \(x = -3\) and away from \(x = 0\).
- For \(x < -3\) the slopes are positive, so the solutions increase towards \(x = -3\).

So we see that:

- The critical point \(x = 0\) is unstable.
- The critical point \(x = 3\) is stable and any solution that is initially positive goes to \(x = 3\) as \(t \to \infty\).
- The critical point \(x = -3\) is also stable and any solution that is initially negative goes to \(x = -3\) as \(t \to \infty\).
Question 5

Consider the logistic equation with a constant harvesting rate \( h \):

\[
\frac{dx}{dt} = x(8 - x) - h.
\]

Find the equilibria (if any) in the cases: \( h = 0 \), \( h = 12 \), \( h = 16 \) and \( h = 25 \). Also discuss the stability of the equilibria.

In each case, if the initial population is \( x = 10 \), discuss what happens to the population as time passes.

The equilibria are the zeroes of the right-hand side, so where:

\[
x(8 - x) - h = 0, \quad 8x - x^2 - h = 0, \quad x^2 - 8x + h = 0, \\
(x - 4)^2 = 16 - h, \quad x = 4 \pm \sqrt{16 - h}.
\]

- When \( h = 0 \), the equilibria are at \( x = 8 \) and \( x = 0 \).
  - When \( x > 8 \), the slope \( x(8 - x) \) is negative, so \( x \) decreases to \( x = 8^+ \).
  - When \( 0 < x < 8 \), the slope \( x(8 - x) \) is positive, so \( x \) increases towards \( x = 8 \) and away from \( x = 0 \).
  - When \( x < 0 \), the slope \( x(8 - x) \) is negative, so \( x \) decreases away from \( x = 0 \).

So \( x = 8 \) is stable and \( x = 0 \) is unstable. In particular if \( x = 10 \) initially, then the population decreases to \( x = 8^+ \) as \( t \to \infty \).

- When \( h = 12 \), the equilibria are at \( x = 6 \) and \( x = 2 \).
  - When \( x > 6 \), the slope \( x(8 - x) - 12 \) is negative, so \( x \) decreases towards \( x = 6 \).
  - When \( 2 < x < 6 \), the slope \( x(8 - x) - 12 \) is positive, so \( x \) increases towards \( x = 6 \) and away from \( x = 0 \).
  - When \( x < 2 \), the slope \( x(8 - x) - 12 \) is negative, so \( x \) decreases away from \( x = 2 \).

So \( x = 6 \) is stable and \( x = 2 \) is unstable.

In particular if \( x = 10 \) initially, then the population decreases to \( x = 6^+ \) as \( t \to \infty \).
• When $h = 16$, there is only one equilibrium at $x = 4$.

Note that in this case the slope is $8x - x^2 - 16 = -(x - 4)^2$.

  – When $x > 4$, the slope $-(x - 4)^2$ is negative, so $x$ decreases towards $x = 4$.
  
  – When $x < 4$, the slope $-(x - 4)^2$ is negative, so $x$ decreases away from $x = 4$, blowing up to minus infinity in finite time.

So $x = 4$ is a saddle, stable from above and unstable from below.

In particular if $x = 10$ initially, then the population decreases to $x = 4^+$ as $t \to \infty$.

Note that the significance of the saddle here is that if there is a slight downward glitch in the population when it is near $x = 4$ (caused by drought, disease, etc.), which temporarily causes the population to drop below $x = 4$, then the population will die out.

• When $h = 25$, the roots of the quadratic are $x = 4 \pm \sqrt{16 - 25} = 4 \pm i\sqrt{3}$.

Note that in this case the slope is $8x - x^2 - 25 = -(x - 4)^2 - 9 < 0$.

So starting from anywhere the slope is always at least as negative as $-9$, so the solution rapidly decreases, and decreases faster and faster as soon as $x$ gets below the level $x = 4$.

Since the slope is more negative that the previous case, $h = 16$, treated above, which blew up to minus infinity in finite time, as soon as $x < 4$, so does this case.

In particular if $x = 10$ initially, then the population decreases to $x = 0$ in finite time (actually for some $t < \frac{10}{9}$, since the slope is at least $-9$).