Matrix Theory and Differential Equations
Homework 3 Solutions, 9/15/6

Question 1
A tank with capacity 100 gallons initially contains 50 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in to the tank at a rate of 3 gallons per minute, is thoroughly mixed and the mixture is pumped out at a rate of 2 gallons per minute. The pumping ceases when the tank is full. How much salt is in the tank then?

Let there be \( x \) pounds of salt in the tank at time \( t \) minutes.

- Since 3 gallons per minute of fluid come in and only 2 gallons go out, there is a net gain of one gallon per minute, so the volume of fluid \( V \) in the tank at time \( t \) (until it overflows) is:
  \[
  V = 50 + t, \quad \text{for } 0 \leq t \leq 50.
  \]

- The inflow rate of salt into the tank is \( 2(3) = 6 \) pounds per minute.

- The outflow rate of salt from the tank is \( \frac{2x}{50 + t} \).

- So the differential equation obeyed by \( x \) is:
  \[
  \frac{dx}{dt} = 6 - \frac{2x}{50 + t}.
  \]

This linear, with the standard form:

\[
\frac{dx}{dt} + \frac{2x}{50 + t} = 6, \quad x(0) = 0.
\]

The integrating factor is:

\[
e^{\int \frac{2}{50+t} dt} = e^{2 \ln(50+t)} = (e^{\ln(50+t)})^2 = (50 + t)^2.
\]
Multiplying both sides of the differential equation by the integrating factor, we get:

\[
(50 + t)^2 \frac{dx}{dt} + 2x(50 + t) = 6(50 + t)^2,
\]

\[
\frac{d}{dt}((50 + t)^2x) = 6(50 + t)^2,
\]

\[
(50 + t)^2x = \int 6(50 + t)^2 dt = 2(50 + t)^3 + C.
\]

Putting \(t = 0\) and \(x = 0\), we get the relation:

\[
0 = 2(50)^3 + C,
\]

\[
C = -250000,
\]

\[
(50 + t)^2x = 2(50 + t)^3 - 250000,
\]

\[
x = 2(50 + t) - \frac{250000}{(50 + t)^2}.
\]

The plot shows an increasing curve, slightly concave down.

As remarked above, the tank overflows at \(t = 50\), when the amount of salt in the tank is:

\[
x(50) = 2(50 + 50) - \frac{250000}{(50 + 50)^2} = 200 - \frac{250000}{10000} = 200 - 25 = 175.
\]

So when full, the tank contains 175 pounds of salt.
Question 2

Tank A is initially full of 100 liters of alcohol. Pure water flows into tank A at 5 liters per minute, is thoroughly mixed and pumped out at the same rate into tank B, which initially contains 50 liters of pure water. There it is again thoroughly mixed and pumped out also at 5 liters per minute. Write and solve differential equations for the amounts of alcohol in each tank. When does the amount of alcohol in tank B peak?

Let there be $x$ liters of alcohol and $y$ liters of alcohol in tanks A and B respectively, at time $t$ minutes. Note that the total volume of fluid in each tank remains constant.

- The inflow rate of alcohol in tank A is 0.
- The outflow rate of alcohol in tank A is $\frac{5x}{100} = \frac{x}{20}$.
- So the differential equation obeyed by $x$ is:
  \[
  \frac{dx}{dt} = 0 - \frac{x}{20} = -\frac{x}{20}, \quad x(0) = 100.
  \]
This is standard exponential decay, with solution $x = 100e^{-\frac{t}{20}}$.

- The inflow rate of alcohol in tank B is $\frac{x}{20}$.
- The outflow rate of alcohol in tank B is $\frac{5y}{50} = \frac{y}{10}$.
- So the differential equation obeyed by $y$ is:
  \[
  \frac{dy}{dt} = \frac{x}{20} - \frac{y}{10}, \quad y(0) = 0.
  \]
We substitute for $x$ in this equation, giving the linear differential equation:

\[
\frac{dy}{dt} + \frac{y}{10} = \frac{100e^{-\frac{t}{20}}}{20} = 5e^{-\frac{t}{20}}, \quad y(0) = 0.
\]
We use either of two solution strategies to solve the differential equation for \( y \):

- We look for a particular solution to the differential equation \( y = y_p \) of the form:

\[
y = Ae^{-\frac{t}{20}}, \text{ with } A \text{ constant.}
\]

Then we need:

\[
\frac{dy}{dt} + \frac{y}{10} = -\frac{A}{20}e^{-\frac{t}{20}} + \frac{A}{10}e^{-\frac{t}{20}} = \frac{A}{20}e^{-\frac{t}{20}} = 5e^{-\frac{t}{20}},
\]

\[
\frac{A}{20} = 5, \quad A = 100,
\]

\[
y_p = 100e^{-\frac{t}{20}}.
\]

Then the full solution to the equation is \( y = y_p + y_h \), where \( y = y_h \) is the general solution of the associated homogeneous equation:

\[
\frac{dy}{dt} + \frac{y}{10} = 0,
\]

\[
\frac{dy}{dt} = -\frac{y}{10}.
\]

This is the standard exponential decay equation, with general solution:

\[
y_h = Ce^{-\frac{t}{20}}.
\]

So the general solution to the equation for \( y \) is:

\[
y = y_p + y_h = 100e^{-\frac{t}{20}} + Ce^{-\frac{t}{10}}.
\]

When \( t = 0 \), we need \( y = 0 \), which gives the relation: \( 0 = 100 + C \), so \( C = -100 \).

So the required solution is:

\[
y = 100(e^{-\frac{t}{20}} - e^{-\frac{t}{10}}).
\]
Alternatively, we use the integrating factor technique. Our equation in standard form is:

\[
\frac{dy}{dt} + \frac{y}{10} = 5e^{-\frac{t}{10}}, \quad y(0) = 0.
\]

The integrating factor is \(e^{\int \frac{1}{10} dt} = e^{\frac{t}{10}}\).

Multiplying both sides of the differential equation by \(e^{\frac{t}{10}}\), we get:

\[
e^{\frac{t}{10}} \frac{dy}{dt} + e^{\frac{t}{10}} \frac{y}{10} = (5e^{-\frac{t}{10}})e^{\frac{t}{10}} = 5e^{-\frac{t}{10}} + \frac{1}{10} = 5e^{\frac{t}{10}},
\]

\[
\frac{d}{dt} e^{\frac{t}{10}}y = 5e^{\frac{t}{10}}, \quad e^{\frac{t}{10}}y = \int 5e^{\frac{t}{10}} dt = 100e^{\frac{t}{10}} + C.
\]

Putting \(t = 0\) and \(y = 0\), we get \(0 = 100 + C\), so \(C = -100\).

Then we get:

\[
y = e^{-\frac{t}{10}} (100e^{\frac{t}{10}} + 100) = 100(e^{-\frac{t}{10}} - e^{-\frac{t}{10}}).
\]

Differentiating our solution to find the critical points, we get:

\[
\frac{dy}{dt} = -5e^{-\frac{t}{10}} + 10e^{-\frac{t}{10}} = 0, \quad \text{when } 5e^{-\frac{t}{10}} = 10e^{\frac{t}{10}},
\]

\[
5e^{\frac{t}{10}} = 10, \quad e^{\frac{t}{10}} = 2, \quad \frac{t}{20} = \ln(2), \quad t = 20 \ln(2).
\]

When \(t = 20 \ln(2)\), we have:

\[
e^{\frac{t}{10}} = 2, \quad e^{-\frac{t}{10}} = \frac{1}{2}, \quad e^{\frac{t}{10}} = (e^{\frac{t}{10}})^2 = 2^2 = 4, \quad e^{-\frac{t}{10}} = \frac{1}{4},
\]

\[
x = 100e^{-\frac{t}{10}} = 100 \left(\frac{1}{2}\right) = 50, \quad y = 100(e^{-\frac{t}{10}} - e^{-\frac{t}{10}}) = 100 \left(\frac{1}{2} - \frac{1}{4}\right) = 50 - 25 = 25.
\]

Note that we can rewrite \(y\) as follows: \(y = 100e^{-\frac{t}{10}}(e^{\frac{t}{10}} - 1)\).

For \(t = 0\), \(y = 0\).

For \(t > 0\), \(e^{\frac{t}{10}} > 1\), so \(y\), being the product of positive factors, is positive.

For \(t \to \infty\), \(y \to 0^+\), since the exponentials \(e^{-\frac{t}{10}}\) and \(e^{-\frac{t}{10}}\) both go to zero.

So somewhere in the interval \((0, \infty)\), \(y\) must have an absolute maximum and this must be at a critical point.

But the only critical point is \(t = 20 \ln(2)\), where \(y = 25\), as just shown.

So this must be the absolute maximum.

So the amount of alcohol in tank B peaks at \(t = 20 \ln(2) = 13.8629\), when the amount of alcohol in tank A is 50 liters and in tank B is 25 liters, so 25 liters of alcohol have passed through the system.
**Question 3**

Consider the differential equation:

\[
dy/dt + 4y = 25 \sin(3t).
\]

- Show that there is a special solution of the differential equation of the form 
  \[ y = A \cos(3t) + B \sin(3t) \] 
  and find the constants \( A \) and \( B \).

  If \( y = A \cos(3t) + B \sin(3t) \), then we have:

  \[
  \frac{dy}{dt} = -3A \sin(3t) + 3B \cos(3t),
  \]

  \[
  \frac{dy}{dt} + 4y = -3A \sin(3t) + 3B \cos(3t) + 4(A \cos(3t) + B \sin(3t))
  \]

  \[
  = (-3A + 4B) \sin(3t) + (3B + 4A) \cos(3t)
  \]

  To satisfy the differential equation, we need this to equal the right-hand side of the differential equation, namely \( 25 \sin(3t) \), which entails that \( A \) and \( B \) obey the following relations:

  \[-3A + 4B = 25, \quad 3B + 4A = 0.\]

  The second equation gives \( B = -\frac{4A}{3} \).

  Substituting into the first equation gives:

  \[-3A + 4 \left(-\frac{4A}{3}\right) = 25,
  \]

  \[-9A - 16A = 75,\]

  \[-25A = 75, \quad A = -3,\]

  \[B = -\frac{4A}{3} = -\frac{4(-3)}{3} = 4.\]

  So the required particular solution is \( y_p = -3 \cos(3t) + 4 \sin(3t) \).
Hence find the general solution and discuss its behavior, as a function of time.

The associated homogeneous equation is:

\[
\frac{dy}{dt} + 4y = 0,
\]

\[
\frac{dy}{dt} = -4y.
\]

This is the standard equation for exponential decay with general solution \(y_h = Ce^{-4t}\).

So the given differential equation has the general solution:

\[
y = y_p + y_h = -3 \cos(3t) + 4 \sin(3t) + Ce^{-4t}.
\]

Find and plot the solution which has the initial condition \(y(0) = 6\).

Putting \(t = 0\) in our general solution gives \(6 = y(0) = -3 + C\), so \(C = 9\) and the required solution is:

\[
y = -3 \cos(3t) + 4 \sin(3t) + 9e^{-4t}.
\]

Plotting the solution for \(t \geq 0\), shows that the solution is effectively indistinguishable from the sinusoidal oscillation \(z = -3 \cos(3t) + 4 \sin(3t)\) of amplitude 5 and period \(\frac{2\pi}{3}\), as soon as \(t \geq 1\).

In the interval \([0, 1]\), there is first a local minimum at \(t = 0.2189\) where \(y = 3.8155\) and then a local maximum at \(t = 0.6866\), where \(y = 5.5179\) and thereafter the \(y\) and \(z\) curves quickly come together.

For example the next local minimum of \(y\) occurs when \(t = 1.7859285\) and \(y = -4.9928827\), to be compared with the corresponding local minimum of \(z\), which occurs at \(t = 1.7852967\), where \(y = -5\).
Question 4

Verify the following equations of differentials are exact and solve them:

- \((3x^2 + y^2)dx + (2xy - 4y^3)dy = 0\).

We have \(Adx + Bdy = 0\), where \(A = 3x^2 + y^2\) and \(B = 2xy - 4y^3 = 0\).

The exactness condition is \(\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}\).

Here we have:

\[
\frac{\partial A}{\partial y} = \frac{\partial}{\partial y}(3x^2 + y^2) = 0 + 2y = 2y,
\]

\[
\frac{\partial B}{\partial x} = \frac{\partial}{\partial x}(2xy - 4y^3) = 2y - 0 = 2y.
\]

So the differential is exact, as required.

To find \(g\), such that \(dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = (3x^2 + y^2)dx + (2xy - 4y^3)dy\), we need to solve both the equations, simultaneously:

\[
\frac{\partial g}{\partial x} = 3x^2 + y^2,
\]

\[
\frac{\partial g}{\partial y} = 2xy - 4y^3.
\]

We solve the first equation by integrating with respect to \(x\) (regarding \(y\) as fixed):

\[
g = \int(3x^2 + y^2)dx = x^3 + y^2x + C(y).
\]

Here the "constant of integration" can depend on \(y\). Then we need:

\[
0 = \frac{\partial g}{\partial y} - (2xy - 4y^3) = \frac{\partial (x^3 + y^2x + C(y))}{\partial y} - 2xy + 4y^3
\]

\[
= 2xy + C'(y) - 2xy + 4y^3 = C'(y) + 4y^3.
\]
So we need:

\[ C''(y) = -4y^3, \]
\[ C = \int -4y^3\,dy = -y^4 + A. \]

Here \( A \) is a constant.
So the required solution is:

\[ g = x^3 + y^2x + C(y) = x^3 + y^2x - y^4 + A. \]

It is easily checked that \( g \) has the required partial derivatives, so we are done and the solution of the differential equation is given implicitly by the equation \( g = 0 \).
In this case, we can more or less solve explicitly for \( y \) in terms of \( x \):

\[ y^4 - y^2x = x^3 + A, \]
\[ \left( y^2 - \frac{x}{2} \right)^2 = A + x^3 + \frac{x^2}{4}, \]
\[ y = \pm \sqrt{\frac{x}{2} \pm \sqrt{A + x^3 + \frac{x^2}{4}}}. \]

To choose the appropriate signs here, we would need an initial condition.

- \((yx^{-2} - x)dx + (y - x^{-1})dy = 0.\)

We have \( Adx + Bdy = 0 \), where \( A = yx^{-2} - x \) and \( B = y - x^{-1} = 0 \).

The exactness condition is \( \frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} \).
Here we have:

\[ \frac{\partial A}{\partial y} = \frac{\partial}{\partial y}(yx^{-2} - x) = x^{-2}, \]
\[ \frac{\partial B}{\partial x} = \frac{\partial}{\partial x}(y - x^{-1}) = x^{-2}. \]

So the differential is exact, as required.
To find \( g \), such that 
\[
dg = \frac{\partial g}{\partial x} \, dx + \frac{\partial g}{\partial y} \, dy = (yx^{-2} - x) \, dx + (y - x^{-1}) \, dy,
\]
we need to solve both the equations, simultaneously:
\[
\frac{\partial g}{\partial x} = yx^{-2} - x, \\
\frac{\partial g}{\partial y} = y - x^{-1}.
\]
We solve the first equation by integrating with respect to \( x \) (regarding \( y \) as fixed):
\[
g = \int (yx^{-2} - x) \, dx = -yx^{-1} - \frac{x^2}{2} + C(y).
\]
Here the "constant of integration" can depend on \( y \).
Then we need:
\[
0 = \frac{\partial g}{\partial y} - (y - x^{-1}) = \frac{\partial}{\partial y} \left( -yx^{-1} - \frac{x^2}{2} + C(y) \right) - y + x^{-1} \\
= -x^{-1} + C'(y) - y + x^{-1} = C'(y) - y.
\]
So we need:
\[
C'(y) = y,
C = \int y \, dy = \frac{y^2}{2} + A.
\]
Here \( A \) is a constant.
So the required solution is:
\[
g = -yx^{-1} - \frac{x^2}{2} + C(y) = -yx^{-1} - \frac{x^2}{2} + \frac{y^2}{2} + A.
\]
It is easily checked that \( g \) has the required partial derivatives, so we are done and the solution of the differential equation is given implicitly by the equation \( g = 0 \).

Again we can more or less solve for \( y \):
\[
y^2 - 2yx^{-1} = x^2 - 2A, \\
(y - x^{-1})^2 = x^2 + x^{-2} - 2A, \\
y = x^{-1} \pm \sqrt{x^2 + x^{-2} - 2A}.
\]
Again we need an initial condition to select the sign of the square root.
**Question 5**

The time rate of change of a population $P$ of rabbits is proportional to the square root of $P$.

Initially $P = 64$ rabbits and $\frac{dP}{dt} = 2$ rabbits per week.

After 50 weeks about how many rabbits will there be?

Explain your answer.

We have, measuring $t$ in weeks:

$$\frac{dP}{dt} = k\sqrt{P}.$$

Putting $t = 0$, we have $P = 64$ and $\frac{dP}{dt} = 2$, so we get:

$$2 = k\sqrt{64} = 8k, \quad k = \frac{2}{8} = \frac{1}{4},$$

$$\frac{dP}{dt} = \frac{1}{4}\sqrt{P},$$

$$\frac{4dP}{P^{\frac{1}{2}}} = dt,$$

$$t = \int 4P^{-\frac{1}{2}}dP = 8P^{\frac{1}{2}} + C.$$

Putting $t = 0$ and $P = 64$, gives:

$$0 = 8\sqrt{64} + C = 8(8) + C = 64 + C,$$

$$C = -64,$$

$$t = 8P^{\frac{1}{2}} - 64,$$

$$t + 64 = 8\sqrt{P},$$

$$P = \frac{1}{64}(t + 64)^2.$$

Putting $t = 50$, we get:

$$P(50) = \frac{1}{64}(50 + 64)^2 = \frac{114^2}{64} = \frac{57^2}{16} = \frac{3249}{16} = 203.0625.$$

So after 50 weeks, the rabbit population is about 203 rabbits.