Matrix Theory and Differential Equations
Homework 2 Solutions, due 9/7/6

Question 1

Consider the differential equation

\[ \frac{dy}{dx} = x - y + 2. \]

- Plot the slope field for the differential equation.
  In particular plot all places with slopes of \(-1, -\frac{1}{2}, 0, \frac{1}{2}\) and 1.

If the slope to be plotted at the point \((x, y)\) is \(m\), we have \(x - y + 2 = m\), so \(y = x + 2 - m\).
This is the equation of a straight line of slope 1 (so forty-five degrees up from the \(x\)-axis) and \(y\)-intercept \(2 - m\).

- When \(m = -1\), the intercept is \((0, 3)\) and we plot downward slopes of \(-1\) along the line \(y = x + 3\).
- When \(m = -\frac{1}{2}\), the intercept is \((0, \frac{5}{2})\) and we plot downward slopes of \(-\frac{1}{2}\) along the line \(y = x + \frac{5}{2}\).
- When \(m = 0\), the intercept is \((0, 2)\) and we plot horizontal slopes of 0 along the line \(y = x + 2\).
- When \(m = \frac{1}{2}\), the intercept is \((0, \frac{3}{2})\) and we plot upward slopes of \(\frac{1}{2}\) along the line \(y = x + \frac{3}{2}\).
- When \(m = 1\), the intercept is \((0, 1)\) and we plot upward slopes of 1 along the line \(y = x + 1\).
  In this case we see that the slopes are all tangent to the line that they are on, so \(y = x + 1\) is a solution of the differential equation.

Check: if \(y = x + 1\), then \(\frac{dy}{dx} = 1\) and \(x - y + 2 = x - (x + 1) + 2 = 1\), so both sides of the differential equation match and the equation holds, as expected.

- When \(m = \frac{3}{2}\), the intercept is \((0, \frac{1}{2})\) and we plot upward slopes of \(\frac{3}{2}\) along the line \(y = x + \frac{1}{2}\).
- When \(m = 2\), the intercept is \((0, 0)\) and we plot upward slopes of 2 along the line \(y = x + 1\).
• Discuss the behavior of the solutions, using your slope field.

  – From the slope field, we see that \( y = x + 1 \) solves the differential equation, as shown above.

  – If we have a solution with initial condition lying above the line \( y = x + 1 \), it everywhere curves concave upwards and eventually approaches the line \( y = x + 1 \) from above, as \( x \) increases to infinity.
  
  The solution has a unique local and global minimum, as the solution crosses the line \( y = x + 2 \).
  
  There are no other critical points and no inflection points.
  
  As \( x \to \pm \infty \), we have \( y \to \infty \).

  – If we have a solution with initial condition lying below the line \( y = x + 1 \), it everywhere curves concave downwards and eventually approaches the line \( y = x + 1 \) from below, as \( x \) increases to infinity.
  
  There are no critical points and no inflection points.
  
  As \( x \to \pm \infty \), we have \( y \to -\infty \).

• One solution \( y \) of the equation is linear in \( x \).

  Find it and plot it on your slope field.

  The solution is \( y = x + 1 \), as described above.

  Alternatively, we put \( y = ax + b \) in the differential equation.

  We need \( \frac{dy}{dx} = a = x - y + 2 = x - ax - b + 2 \).

  Collecting terms we need: \( 0 = x(1 - a) - b - a + 2 \).

  Since this equation is required to hold for more than one \( x \)-value, we need its coefficient to be 0, so we need \( a = 1 \).

  Then the constant term must be zero also, so \( b = -a + 2 = -1 = 2 = 1 \).

  So \( y = x + 1 \), as expected. Now we may find the general solution as follows.

  Put \( y = z + x + 1 \), so \( z = y - x - 1 \).

  Then if \( y \) obeys the differential equation, we have:

  \[
  \frac{dz}{dx} = \frac{d}{dx}(y-x-1) = \frac{dy}{dx} - 1 = x-y+2-1 = x-y+1 = -(y-x-1) = -z.
  \]

  Conversely if \( z \) obeys the equation, \( \frac{dz}{dx} = -z \), we see that \( y \) necessarily obeys the given differential equation.

  The equation \( \frac{dz}{dx} = -z \) is a standard equation for exponential decay, with general solution \( z = Ae^{-x} \), with \( A \) constant.
With $z = Ae^{-x}$, we have $y = z + x + 1 = Ae^{-x} + x + 1$ as the general solution of the given differential equation.

- If $A = 0$, we recover the linear solution $y = x + 1$.

- If $A > 0$, then the solution lies above the line $y = x + 1$, since the extra term $Ae^{-x}$ is always positive.
  
  We have: $\frac{dy}{dx} = -Ae^{-x} + 1 = 0$, when $Ae^{-x} = 1$, so $e^x = A$ and $x = \ln(A)$.
  
  Then $y = Ae^{-x} + x + 1 = 1 + x + 1 = x + 2$.
  
  So the only critical point lies on the line $y = x + 2$, as found above.
  
  If the critical point has $x$-co-ordinate $x_0$, then the relation $A = e^{x_0}$ determines the parameter $A$.
  
  We have: $\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx}(-Ae^{-x} + 1) = Ae^{-x} > 0$.
  
  So the curve is everywhere concave up, so the point $(\ln(A), \ln(A) + 2)$ is a local and global minimum, as predicted above.
  
  As $x \to \infty$, $Ae^{-x} \to 0^+$, so the curve approaches the line $y = x + 1$ from above.
  
  As $x \to -\infty$, $Ae^{-x} \to \infty$, so $Ae^{-x} + x + 1 \to \infty$ also (since the exponential dominates any polynomial in $x$), so the curve goes to infinity.
  
  So the solution decreases from infinity, always concave up, reaches its minimum and then curves up towards the line $y = x + 1$ as $x \to \infty$.

- If $A < 0$, then the solution lies above the line $y = x + 1$, since the extra term $Ae^{-x}$ is always negative.

  We have: $\frac{dy}{dx} = -Ae^{-x} + 1 = 0$, when $Ae^{-x} = 1$, which is impossible, since the left-hand side of this equation is negative.

  So there are no critical points.

  We have: $\frac{d^2y}{dx^2} = Ae^{-x} < 0$, so the curve is everywhere concave down.

  As $x \to \infty$, $Ae^{-x} \to 0^-$, so the curve approaches the line $y = x + 1$ from below.

  As $x \to -\infty$, $Ae^{-x} \to -\infty$, so $Ae^{-x} + x + 1 \to -\infty$, so the curve goes to minus infinity.

  So the solution increases from minus infinity, always concave down and curves up towards the line $y = x + 1$ as $x \to \infty$.
Alternatively, we can use the integrating factor technique. We write the differential equation in the standard form:

\[ \frac{dy}{dx} + y = x + 2. \]

The integrating factor is \( J(x) = e^{\int 1 \, dx} = e^x \).
Then multiplying by the integrating factor, the equation becomes:

\[ e^x \frac{dy}{dx} + e^x y = e^x(x + 2), \]
\[ \frac{d}{dx}(e^x y) = xe^x + 2e^x, \]
\[ e^x y = \int (xe^x + 2e^x) \, dx = xe^x - e^x + 2e^x + A = e^x(x + 1) + A. \]

Dividing by \( e^x \), using \( \frac{1}{e^x} = e^{-x} \) gives:
\[ y = x + 1 + Ae^{-x}. \]

This agrees with our previous solution.
Question 2

A population $P$ hundred deer at time $t$ years is governed by the logistic differential equation:

$$\frac{dP}{dt} = \frac{P}{2} - \frac{P^2}{5}.$$

- Plot the slope field for the differential equation including the slopes with $P = 0, 1, 2, 3, 4$ and $5$.

We plot the same slope for a given $P$ at every $t$-value.

Note that $\frac{P}{2} - \frac{P^2}{5} = \frac{P}{10}(5 - 2P)$.

- For the unphysical case that $P < 0$, the slopes plotted are negative and get more negative as $P$ decreases.
- For $P = 0$ and for $P = \frac{5}{2}$, the slope is everywhere 0 and therefore $P = 0$ and $P = \frac{5}{2}$ for all time are solutions.
- For $P < 0$, the slopes plotted are negative and get more negative as $P$ decreases.
- For $P = 1$ and $P = 2$, the slopes are positive: $\frac{3}{10}$ and $\frac{1}{5}$, respectively.
- For $P = 3, P = 4$ and $P = 5$, the slopes are negative: $-\frac{3}{10}$ and $-\frac{6}{5}$ and $-\frac{5}{2}$ respectively.

- Discuss the nature of the solutions.

We have a standard logistic equation. Let the initial population be $P_0$.

- If $P_0 = 0$, then the population is zero always.
- If $P_0 = \frac{5}{2}$, then the population is $\frac{5}{2}$ always.
- All solutions with $P_0 > \frac{5}{2}$ are concave up, approaching $P = \frac{5}{2}$ from above as $t \to \infty$. Going backward in time, they blow up in finite time.
- All (unphysical) solutions with $P_0 < 0$ curve are concave down, approaching $P = 0^-$ from below as $t \to -\infty$.

In the future they go to minus infinity in finite time.
- All solutions with $0 < P_0 < \frac{5}{2}$ approach $0^+$ as $t \to -\infty$ and approach $\frac{5}{2}^-$ as $t \to \infty$. They are strictly increasing for all time, and have a unique inflection point as they cross the line $P = \frac{5}{4}$.

Before the inflection point the graph is concave up, after it is concave down.
• What is the deer population when the rate of population growth is largest and what is that maximum rate of change? Explain your answer.

The maximum rate of change occurs when the quadratic \( \frac{P}{10}(5 - 2P) \) reaches its maximum value. This is at the point half way between the two intercepts at \( P = 0 \) and \( P = \frac{5}{2} \), so is at \( P = \frac{5}{4} \), when the rate of change is:

\[
\frac{5}{4} \left( \frac{5}{2} \right) = \frac{5}{40} \left( \frac{5}{2} \right) = \frac{5}{16}.
\]

The units here are in hundreds of deer per year, so the maximum rate of change is \( \frac{500}{16} = 31.25 \) deer per year.

• Solve the equation exactly with the initial condition \( P = 1 \) and plot it on your slope field.

We have:

\[
\frac{dP}{dt} = \frac{P}{2} - \frac{P^2}{5} = \frac{P}{10}(5 - 2P),
\]

\[
\frac{dt}{dP} = \frac{10}{P(5 - 2P)} = \frac{A}{P} + \frac{B}{5 - 2P},
\]

\[
A(5 - 2P) + BP = 10.
\]

Putting \( P = 0 \), we get \( 5A = 10 \), so \( A = 2 \).

Then \( 2(5 - 2P) + BP = 10 \), so \( 10 + (B - 4)P = 10 \), so \( (B - 4)P = 0 \), so \( B = 4 \).

So we have:

\[
\frac{dt}{dP} = \frac{2}{P} + \frac{4}{5 - 2P},
\]

\[
t = \int \left( \frac{2}{P} + \frac{4}{5 - 2P} \right) dP = 2 \ln(P) - 2 \ln(A(5 - 2P)) = 2 \ln \left( \frac{P}{A(5 - 2P)} \right).
\]

Putting \( t = 0 \) and \( P = 1 \), we get: \( 0 = 2 \ln \left( \frac{1}{3A} \right) \), so \( A = \frac{1}{3} \).

Then we have:

\[
t = 2 \ln \left( \frac{P}{\left( \frac{1}{3} \right)(5 - 2P)} \right), \quad \frac{t}{2} = \ln \left( \frac{3P}{5 - 2P} \right), \quad e^{\frac{t}{2}} = \frac{3P}{5 - 2P}.
\]

\[
(5 - 2P)e^{\frac{t}{2}} = 3P, \quad P(2e^{\frac{t}{2}} + 3) = 5e^{\frac{t}{2}}, \quad P = \frac{5e^{\frac{t}{2}}}{2e^{\frac{t}{2}} + 3} = \frac{5}{2 + 3e^{-\frac{t}{2}}}.
\]
Alternatively, we can get the solution as follows:

\[ Q = \frac{1}{P}, \]

\[
\frac{dQ}{dt} = -\frac{1}{P^2} \frac{dP}{dt} = -\frac{1}{P^2} \left( \frac{P}{2} - \frac{P^2}{5} \right) \\
= -\frac{1}{2P} + \frac{1}{5} = \frac{1}{5} - \frac{Q}{2} = -\frac{1}{2} \left( Q - \frac{2}{5} \right).
\]

Put \( Q = R + \frac{2}{5}. \)

Then we have:

\[
\frac{dR}{dt} = \frac{dQ}{dt} = -\frac{1}{2} \left( Q - \frac{2}{5} \right) = -\frac{R}{2},
\]

\[ R = Ae^{-\frac{t}{2}}, \]

\[ Q = \frac{2}{5} + R = \frac{2}{5} + Ae^{-\frac{t}{2}}. \]

When \( t = 0 \), we have \( P = 1 \), so \( Q = 1 \) also, so \( 1 = \frac{2}{5} + A \), so \( A = \frac{3}{5} \).

So we have:

\[ Q = \frac{1}{5} (2 + 3e^{-\frac{t}{2}}) = P^{-1}, \]

\[ P = \frac{5}{2 + 3e^{-\frac{t}{2}}}. \]

The plot is a standard logistic curve, beginning at \( P = 0^+ \) as \( t \to -\infty \), increasing and concave up, passing through the initial point \((0, 1)\), crossing the line \( P = \frac{1}{2} \) when \( e^{-\frac{t}{2}} = \frac{2}{3} \), so at \( t = \ln \left( \frac{2}{3} \right) \) at which point it inflects to concave down and then approaches \( P = \frac{5}{2} \) from below, as \( t \to \infty \).

The curve is everywhere increasing.
Question 3

Solve the following differential equations, with the given initial condition and for each determine the behavior of the solution, with a sketch:

\[ \frac{dy}{dx} = \frac{y^2 + 1}{x}, \quad y = \frac{3}{4} \text{ when } x = \frac{5}{4}. \]

This is separable.

We separate and integrate:

\[
\int \frac{y \, dy}{y^2 + 1} = \int \frac{dx}{x},
\]

\[
\frac{1}{2} \ln(y^2 + 1) = \ln(Ax),
\]

\[
\ln(y^2 + 1) = 2 \ln(Ax) = \ln((Ax)^2),
\]

\[
y^2 + 1 = A^2x^2.
\]

Putting \( y = \frac{3}{4} \) and \( x = \frac{5}{4} \) gives:

\[
\frac{9}{16} + 1 = \frac{25}{16} = \frac{25}{16}A^2, \text{ so } A^2 = 1 \text{ and we have:}
\]

\[
y^2 + 1 = x^2,
\]

\[
y = \sqrt{x^2 - 1}.
\]

The positive square root is taken, since \( y \) is initially positive.

The solution curve lies in the first quadrant and is the upper part of the right branch of the rectangular hyperbola \( x^2 - y^2 = 1 \), starting at the point \((1, 0)\), where the slope is infinite and henceforth curving concave down, strictly increasing and asymptotically approaching the slant asymptote \( y = x \) from below as \( x \to \infty \).
• \( \frac{dy}{dx} = 6e^{2x-y}, \ y = 2 \) when \( x = 0 \).

This is separable, since \( e^{2x-y} = e^{2x}e^{-y} \).

We separate and integrate:

\[
\frac{dy}{dx} = 6e^{2x}e^{-y},
\]

\[
e^y dy = 6e^{2x} dx,
\]

\[
\int e^y dy = \int 6e^{2x} dx,
\]

\[
e^y = 3e^{2x} + C.
\]

Putting \( x = 0 \) and \( y = 2 \), we get \( e^2 = 3 + C \), so \( C = e^2 - 3 \).

Then we have the solution as:

\[
y = \ln(e^2 - 3 + 3e^{2x}).
\]

Note that when \( x \) is large and positive, the term \( 3e^{2x} \) in the logarithm dominates, so \( y \) is approximately \( \ln(3e^{2x}) = 2x + \ln(3) = 2x + 1.098612289 \), whereas, when \( x \) is large and negative the \( e^2 - 3 \) term dominates and \( y \) is approximately the constant \( \ln(e^2 - 3) = 1.479114192 \). The plot is a simple increasing concave up curve, starting from its horizontal asymptote at \( y = \ln(e^2 - 3) \), which it approaches from above as \( x \to -\infty \) and passing through the initial point \((0, 2)\) as it crosses the \( y \)-axis and then curving up towards the slant line asymptote \( y = 2x + \ln(3) \) which it approaches from above as \( x \to \infty \).

• \( \frac{x dy}{dx} + 5y = 7x^2, \ y = 1 \) when \( x = 2 \).

Looking at both sides of the equation, we see that a quadratic solution might exist.

So we put \( y = ax^2 + bx + c \) giving the equation:

\[
7x^2 = x \frac{dy}{dx} + 5y = x \frac{d}{dx}(ax^2 + bx + c) + 5(ax^2 + bx + c)
\]

\[
= x(2ax + b) + 5(ax^2 + bx + c) = 7ax^2 + 6bx + 5c.
\]

So \( a = 1, b = 0, c = 0 \) will do and we have a solution \( y = x^2 \).
Next put \( y = x^2 + z \).
Then we need:

\[
7x^2 = x \frac{dy}{dx} + 5y = x \frac{d}{dx}(z + x^2) + 5(z + x^2) = x \frac{dz}{dx} + 5z + 7x^2,
\]

\[
\frac{dz}{dx} + 5z = 0, \quad \frac{dz}{z} = -\frac{5dx}{x},
\]

\[
\int \frac{dz}{z} = -5 \int \frac{dx}{x}, \quad \ln(z) = \ln(Ax^{-5}),
\]

\[
z = Ax^{-5}, \quad y = x^2 + z = x^2 + Ax^{-5}.
\]

Putting \( y = 1 \) and \( x = 2 \), we get:

\[
1 = 4 + \frac{A}{32}, \quad \frac{A}{32} = -3, \quad A = -96,
\]

\[
y = x^2 - \frac{96}{x^5}.
\]

The curve increases for all \( x > 0 \), starting at \(-\infty \) as \( x \to 0^+ \).

It is initially concave down. It crosses the \( x \)-axis at \((96\frac{2}{3}, 0) = (1.9195, 0)\)
and inflects at \(\left(1440\frac{1}{2}, \left(\frac{14}{12}\right) 1440\frac{1}{2}\right) = (2.826, 7.4546)\).

Then it is concave up and approaches the parabola \( y = x^2 \) from below as \( x \to \infty \).

Alternatively we solve using the integrating factor technique.
We first put the equation in standard form, by dividing through by \( x \) to make
the coefficient of \( \frac{dy}{dx} \) one: \( \frac{dy}{dx} + \frac{5y}{x} = 7x \).

So the integrating factor is \( e^{\int \frac{5}{x} dx} = e^{5 \ln(x)} = x^5 \).

Then, multiplying the equation by \( x^5 \), we get:

\[
x^5 \frac{dy}{dx} + 5x^4 y = \frac{d}{dx}(yx^5) = 7x(x^5) = 7x^6
\]

\[
yx^5 = \int 7x^6 dx = x^7 + C.
\]

Putting \( x = 2 \) and \( y = 1 \) gives:

\[
32 = 128 + C, \quad C = -96,
\]

\[
yx^5 = x^7 - 96,
\]

\[
y = x^2 - \frac{96}{x^5}.
\]
**Question 4**

Show that for any value of the constants $A$ and $B$, the function \( y = A \cos(2t) + B \sin(2t) \) obeys the differential equation: \( \frac{d^2y}{dt^2} = -4y \).

Find the solution that obeys the initial conditions \( y(0) = 3 \), \( y'(0) = 4 \) and discuss the behavior of the solution, with a plot.

\[
y = A \cos(2t) + B \sin(2t),
\]

\[
\frac{dy}{dt} = 2B \cos(2t) - 2A \sin(2t)
\]

\[
\frac{d^2y}{dt^2} = -4A \cos(2t) - 4B \sin(2t) = -4y.
\]

Putting \( y = 3 \) and \( t = 0 \) in the equation for \( y \) gives \( 3 = A \).

Putting \( \frac{dy}{dt} = 4 \) and \( t = 0 \) in the equation for \( \frac{dy}{dt} \) gives \( 4 = 2B \), so \( B = 2 \). so the required solution is:

\[
y = 2 \sin(2t) + 3 \cos(2t).
\]

The plot is sinusoidal, of period \( \pi = 3.14159 \).

The amplitude is \( \sqrt{13} = 3.60555 \).

We can rewrite \( y \) as:

\[
y = \sqrt{13} \sin(2t + \alpha) = \sqrt{13} \cos(\alpha) \sin(2t) + \sqrt{13} \sin(\alpha) \cos(2t),
\]

\[
\cos(\alpha) = \frac{2}{\sqrt{13}}, \quad \sin(\alpha) = \frac{3}{\sqrt{13}},
\]

\[
\alpha = \arctan \left( \frac{2}{3} \right) = 0.98279 \text{ radians.}
\]

Then the maxima, of value \( y = \sqrt{13} \), are at \( t = t^n_+ \), where \( t^n_+ = \frac{1}{4}((4n+1)\pi - 2\alpha) \) and the minima, of value \( y = -\sqrt{13} \) are at \( t = t^n_- \), where \( t^n_- = \frac{1}{4}((4n+3)\pi - 2\alpha) \), for \( n \) any integer.

The first maximum is at \( t = t^0_+ = 0.294 \) and the first minimum is at \( t = t^0_- = 1.865 \).

The inflection points are at the points where \( y'' = 0 \), so also where \( y = 0 \), so the points with \( t = t_m \), where \( t_m = \frac{1}{2}(m\pi - \alpha) \), where \( m \) is integral.

For each integer \( n \), in the intervals \([t_{2n}, t_{2n+1}]\), the curve is concave down, with a maximum at its midpoint \( t = t^n_+ \), whereas in the intervals \([t_{2n+1}, t_{2n+2}]\), the curve is concave up, with a minimum at its midpoint \( t = t^n_- \).
**Question 5**

A tank contains 1000 liters of water with 100 kilograms of salt dissolved in it. Pure water is poured into the tank at a rate of 5 liters per second, is stirred thoroughly and the mixture is pumped out at the same rate. Write and solve a differential equation for the amount of salt in the tank at time \( t \) seconds after the pouring begins.

At what time will there be only one kilogram of salt remaining in the tank?

At time \( t \), let there be \( X \) kilograms of dissolved salt in the tank.

- Since the volume of water in the tank is held constant at 1000 liters, there are \( \frac{X}{1000} \) kilos of salt per liter.

- So the rate of removal of salt from the tank is \( \frac{5X}{1000} = \frac{X}{200} \) kilos per second.

- Since no salt is coming in, the differential equation governing \( X \) is:

\[
\frac{dX}{dt} = -\frac{X}{200}, \quad X(0) = 100.
\]

The equation is a standard exponential decay, with solution:

\[
X = 100e^{-\frac{t}{200}}.
\]

There is one kilogram of salt remaining in the tank when:

\[
1 = 100e^{-\frac{100}{200}} = \frac{100}{e^{\frac{100}{200}}},
\]

\[
e^{\frac{100}{200}} = 100,
\]

\[
\frac{t}{200} = \ln(100),
\]

\[
t = 200 \ln(100) = 921.034.
\]

So after 15 minutes and 21.034 seconds, there will be one kilogram of salt remaining in the tank.