Matrix Theory and Differential Equations
Homework 10 Solutions, due 11/2/6

Question 1

Find a basis for the row space and a basis for the column space of the following matrix:

\[
\begin{bmatrix}
1 & -1 & -2 & -4 \\
2 & 4 & 8 & -1 \\
2 & 3 & 6 & -2 \\
3 & 3 & 6 & -5
\end{bmatrix}
\]

To find a basis for the row space we row reduce the given matrix:

\[
\begin{bmatrix}
1 & -1 & -2 & -4 \\
2 & 4 & 8 & -1 \\
2 & 3 & 6 & -2 \\
3 & 3 & 6 & -5
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -1 & -2 & -4 \\
0 & 6 & 12 & 7 \\
0 & 5 & 10 & 6 \\
0 & 6 & 12 & 7
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & -3 \\
0 & 1 & 2 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Here we did the row operations:

\[R_2 \rightarrow -2R_1 + R_2,\ R_3 \rightarrow -2R_1 + R_3,\ R_4 \rightarrow -3R_1 + R_4,\ R_4 \rightarrow -R_2 + R_4,\]
\[R_2 \rightarrow -R_3 + R_2,\ R_1 \rightarrow R_2 + R_1,\ R_3 \rightarrow -5R_2 + R_3,\ R_1 \rightarrow 3R_3 + R_1,\]
\[R_2 \rightarrow -R_3 + R_2.
\]

Then a basis for the row space is the set of non-zero rows of the reduced matrix so is \{[1, 0, 0, 0], [0, 1, 2, 0], [0, 0, 0, 1]\}.
To find a basis for the column space we row reduce the transpose of the given matrix:

\[
\begin{bmatrix}
1 & 2 & 2 & 3 \\
-1 & 4 & 3 & 3 \\
-2 & 8 & 6 & 6 \\
-4 & -1 & -2 & -5
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 2 & 3 \\
0 & 6 & 5 & 6 \\
0 & 12 & 10 & 12 \\
0 & 7 & 6 & 7
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 2 & 3 \\
0 & 7 & 6 & 7 \\
0 & 6 & 5 & 6 \\
0 & 0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 0 & 3 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Here we did the row operations:

\(R_2 \rightarrow R_1 + R_2, \ R_3 \rightarrow 2R_1 + R_3, \ R_4 \rightarrow 4R_1 + R_4, \ R_2 \leftrightarrow R_4, \ R_3 \leftrightarrow R_4, \)

\(R_4 \rightarrow -2R_3 + R_4, \ R_2 \rightarrow -R_3 + R_2, \ R_3 \rightarrow -6R_2 + R_3, \ R_1 \rightarrow 2R_3 + R_1, \)

\(R_2 \rightarrow R_3 + R_2, \ R_3 \rightarrow -R_3, \ R_1 \rightarrow -2R_2 + R_1.\)

Then a basis for the column space is the set of non-zero rows of the reduced matrix so is \(\{[1, 0, 0, 1], [0, 1, 0, 1], [0, 0, 1, 0]\}\) (where these vectors are written as columns).

Alternatively, we know from work done in class that the columns associated to the pivot variables in the row reduction of the original matrix form a basis for the column space.

So here the pivot variables lie in the first second and fourth columns.

So the following vectors (written as columns) form a basis for the column space:

\(\{[1, 2, 2, 3], [-1, 4, 3, 3], [-4, -1, -2, -5]\}\).
Question 2

Show that the triangle $ABC$ with the following vertices is right-angled and determine its angles and area:

$A = [2, 4, 0, 2], B = [3, 1, 5, 3], C = [0, 2, 3, 1]$.

We have:

$a = BC = C - B = [0, 2, 3, 1] - [3, 1, 5, 3] = [-3, 1, -2, -2],$

$b = CA = A - C = [2, 4, 0, 2] - [0, 2, 3, 1] = [2, 2, -3, 1],$

$c = AB = B - A = [3, 1, 5, 3] - [2, 4, 0, 2] = [1, -3, 5, 1].$

Then the side lengths squared for the triangle are:

$a^2 = a \cdot a = [-3, 1, -2, -2][-3, 1, -2, -2] = 9 + 1 + 4 + 4 = 18,$

$b^2 = b \cdot b = [2, 2, -3, 1][2, 2, -3, 1] = 4 + 4 + 9 + 1 = 18,$

$c^2 = c \cdot c = [1, -3, 5, 1][1, -3, 5, 1] = 1 + 9 + 25 + 1 = 36.$

So $c = 6$ and $a = b = \sqrt{18} = 3\sqrt{2}$.

Since $36 = c^2 = a^2 + b^2 = 18 + 18$, the triangle is right-angled, with the right-angle at the vertex $C$.

Since $a = b$, the triangle is isosceles and each of the other two angles is $\frac{\pi}{4}$ radians, or 45 degrees.

Then the area of the triangle is $\frac{1}{2}ab = \frac{1}{2}a^2 = 9$ units of area.
Question 3

Find an orthonormal basis for the subspace of \( \mathbb{R}^4 \) orthogonal to the subspace spanned by the vectors \([2, 3, 1, -4]\) and \([1, 2, -1, 3]\).

A vector \([t, x, y, z]\) lies in the orthogonal space if it is perpendicular to each of the vectors \([2, 3, 1, -4]\) and \([1, 2, -1, 3]\). So it obeys the equations:

\[
t + 2x - y + 3z = 0, \quad 2t + 3x + y - 4z = 0.
\]

We reduce the coefficient matrix:

\[
\begin{pmatrix}
1 & 2 & -1 & 3 \\
2 & 3 & 1 & -4
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & -1 & 3 \\
0 & -1 & 3 & -10
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 5 & -17 \\
0 & 1 & -3 & 10
\end{pmatrix}
\]

Here we did the row operations: \(R_2 \rightarrow -2R_1 + R_2, R_1 \rightarrow 2R_2 + R_1\) and \(R_2 \rightarrow -R_2\).

The reduced system is \(t + 5y - 17z = 0\) and \(x - 3y + 10z = 0\), with general solution:

\([t, x, y, z] = [-5s + 17t, 3s - 10t, s, t] = s[-5, 3, 1, 0] + t[17, -10, 0, 1]\).

So the orthogonal subspace has basis the vectors \(A = [-5, 3, 1, 0]\) and \(B = [17, -10, 0, 1]\).

To finish we need an orthonormal basis for this space.

We may take the unit vector in the direction of \(A\) to be one of the vectors in the basis.

Then we look for a vector of the form \(C = sA + tB\) for the direction of the second vector. This must be perpendicular to \(A\), so we need:

\[0 = C \cdot A = sA \cdot A + tB \cdot A = s(25 + 9 + 1 + 0) + t(-85 - 30 + 0 + 0) = 35s - 115t = 5(7s - 23t).\]

We take \(s = 23\) and \(t = 7\) giving the vector:

\(C = 23[-5, 3, 1, 0] + 7[17, -10, 0, 1] = [-115 + 119, 69 - 70, 23, 7] = [4, -1, 23, 7]\).

We check that this is orthogonal to \(A\):

\([4, -1, 23, 7] \cdot [-5, 3, 1, 0] = -20 - 3 + 23 + 0 = 0\).

We have:

\(|A|^2 = A \cdot A = 25 + 9 + 1 + 0 = 35, \quad |C|^2 = C \cdot C = 16 + 1 + 529 + 49 = 595,\)

Then a required orthonormal basis is:

\[
\left\{ \frac{A}{|A|}, \frac{C}{|C|} \right\} = \left\{ \frac{1}{\sqrt{35}}[-5, 3, 1, 0], \frac{1}{\sqrt{595}}[4, -1, 23, 7] \right\}.
\]
Question 4

Let $\mathcal{F}$ be the vector space of continuous functions on the real line. Which of the following subsets of $\mathcal{F}$ are subspaces? Explain your answers.

- The set of all functions in $\mathcal{F}$ that are everywhere non-negative.

  This is not a subspace since the constant function 1 lies in the space, but its negative does not.

- The set $\mathcal{A}$ of all functions $f(x)$ in $\mathcal{F}$ that vanish when $x$ is any integer.

  The functions $f(x)$ of $\mathcal{A}$ obey $f(n) = 0$, for any integer $n$.

  - If $s$ is any scalar and $f(x)$ is in $\mathcal{A}$, then $(sf)(n) = sf(n) = s(0) = 0$, for any integer $n$, so $(sf)(x)$ is in $\mathcal{A}$.

  - If functions $f(x)$ and $g(x)$ are in $\mathcal{A}$, then $(f + g)(n) = f(n) + g(n) = 0 + 0 = 0$, for any integer $n$, so $(f + g)(x)$ is in $\mathcal{A}$.

  So $\mathcal{A}$ is closed under scalar multiplication and addition, so is a subspace.

- The set $\mathcal{B}$ of all functions $f(x)$ in $\mathcal{F}$ that are odd: $f(x) = -f(-x)$, for any real $x$.

  The functions $f(x)$ of $\mathcal{B}$ obey $f(x) + f(-x) = 0$, for any real $x$.

  - If $s$ is any scalar and $f(x)$ is in $\mathcal{B}$, then $(sf)(x) + (sf)(-x) = sf(x) + sf(-x) = s(f(x) + f(-x)) = s(0) = 0$, for any real $x$, so $(sf)(x)$ is in $\mathcal{B}$.

  - If functions $f(x)$ and $g(x)$ are in $\mathcal{B}$, then $(f + g)(x) + (f + g)(-x) = f(x) + g(x) + f(-x) + g(-x) = (f(x) + f(-x)) + (g(x) + g(-x)) = 0 + 0 = 0$, for any real $x$, so $(f + g)(x)$ is in $\mathcal{B}$.

  So $\mathcal{B}$ is closed under scalar multiplication and addition, so is a subspace.
• The set $\mathcal{C}$ of all functions $f(x)$ in $\mathcal{F}$ that are even: $f(x) = f(-x)$, for any real $x$.

The functions $f(x)$ of $\mathcal{C}$ obey $f(x) = f(-x)$, for any real $x$.

  – If $s$ is any scalar and $f(x)$ is in $\mathcal{C}$, then $(sf)(x) = sf(x) = sf(-x) = (sf)(-x)$, for any real $x$, so $(sf)(x)$ is in $\mathcal{C}$.

  – If functions $f(x)$ and $g(x)$ are in $\mathcal{C}$, then $(f + g)(x) = f(x) + g(x) = f(-x) + g(-x) = (f + g)(-x)$, for any real $x$, so $(f + g)(x)$ is in $\mathcal{C}$.

So $\mathcal{C}$ is closed under scalar multiplication and addition, so is a subspace.

• The set $\mathcal{D}$ of all functions $f(x)$ in $\mathcal{F}$ that are periodic with period 1: $f(x + 1) = f(x)$, for any real $x$.

The functions $f(x)$ of $\mathcal{D}$ obey $f(x + 1) = f(x)$, for any real $x$.

  – If $s$ is any scalar and $f(x)$ is in $\mathcal{D}$, then $(sf)(x + 1) = sf(x + 1) = sf(x) = (sf)(x)$, for any real $x$, so $(sf)(x)$ is in $\mathcal{D}$.

  – If functions $f(x)$ and $g(x)$ are in $\mathcal{D}$, then $(f + g)(x + 1) = f(x + 1) + g(x + 1) = f(x) + g(x) = (f + g)(x)$, for any real $x$, so $(f + g)(x)$ is in $\mathcal{D}$.

So $\mathcal{D}$ is closed under scalar multiplication and addition, so is a subspace.
Question 5

Find, with proof, a basis for the solution space of the following equations:

- $y'' - 4y' + 3y = 0$.
  Put $y = e^{mt}$ with $m$ constant.
  Then $y' = me^{mt}$ and $y'' = m^2 e^{mt}$, so the differential equation becomes:
  
  $$0 = e^{mt}(m^2 - 4m + 3) = e^{mt}(m - 3)(m - 1).$$
  
  So $m = 1$, or $m = 3$ and the general solution is $y = Ae^t + Be^{3t}$.
  We claim that a basis is $\{e^t, e^{3t}\}$.
  This set clearly spans the solution space so we just need to verify linear independence.
  Suppose that $Ae^t + Be^{3t} = 0$; differentiating with respect to $t$, we get $Ae^t + 3Be^{3t} = 0$.
  Putting $t = 0$ in each of these relations, we get $A + B = 0$ and $A + 3B = 0$.
  Subtracting these formulas gives $2B = 0$, so $B = 0$.
  Then $A = -B = -0 = 0$.
  So $A = B = 0$ and the set $\{e^t, e^{3t}\}$ is linearly independent and spans the solution space, so forms a basis and we are done.

- $y'' + 9y = 0$.
  Put $y = e^{mt}$ with $m$ constant.
  Then $y' = me^{mt}$ and $y'' = m^2 e^{mt}$, so the differential equation becomes:
  
  $$0 = e^{mt}(m^2 + 9) = e^{mt}(m - 3i)(m + 3i).$$
  
  So $m = \pm 3i$, and the general solution is:
  
  $$y = Ae^{3it} + Be^{-3it} = A(\cos(3t) + i \sin(3t)) + B(\cos(3t) - i \sin(3t))$$
  
  $$= C \cos(3t) + D \sin(3t), \quad C = A + B, \quad D = iA - iB.$$ 
  
  We claim that a basis is $\{\cos(3t), \sin(3t)\}$.
  This set clearly spans the solution space so we just need to verify linear independence.
  Suppose that $C \cos(3t) + D \sin(3t) = 0$.
  Putting $t = 0$, we get $C = 0$, since $\cos(0) = 1$ and $\sin(0) = 0$.
  Putting $t = \frac{\pi}{6}$, we get $D = 0$, since $\cos \left( \frac{\pi}{2} \right) = 0$ and $\sin \left( \frac{\pi}{2} \right) = 1$.
  So $C = D = 0$ and the set $\{\cos(3t), \sin(3t)\}$ is linearly independent and spans the solution space, so forms a basis and we are done.