Question 1

Let \( \mathbf{F} = [x^2 - y^2, xy + 2x^2] \). Let \( A = [4, 2], B = [3, -1] \) and \( C = [5, -3] \).

- Find the flux of \( \mathbf{F} \) outside the triangle \( ABC \).

We have:

\[
\text{div}(\mathbf{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \frac{\partial (x^2 - y^2)}{\partial x} + \frac{\partial (xy + 2x^2)}{\partial y} = 2x + x = 3x.
\]

Then by the divergence theorem, we have, for the region \( R \) the interior of the triangle \( ABC \), the flux of \( \mathbf{F} \) out of the region \( R \) is: (with \( \mathbf{N} \) the outward pointing unit normal and \( s \) the parametrization by length along the curve):

\[
\int_{\partial R} \mathbf{F} \cdot \mathbf{N} ds = \iint_R \text{div} (\mathbf{F}) dA
\]

\[
= \iint_R 3x dA = [3, 0]. \iint_R [x, y] dA = [3, 0]. \mathbf{M}.
\]

Here \( \mathbf{M} \) is the total moment vector of the triangle about the origin. Then we have, if \( \mathbf{C} \) is the centroid of the triangle and \( \mathcal{A} \) is its area:

\[
\mathbf{M} = \mathbf{C} \mathcal{A}.
\]

For the centroid of the triangle, \( ABC \), we use the standard formula:

\[
\mathbf{C} = \frac{1}{3}(\mathbf{A} + \mathbf{B} + \mathbf{C})
\]

\[
= \frac{1}{3}([4, 2] + [3, -1] + [5, -3]) = \frac{1}{3}[12, -2] = \left[4, -\frac{2}{3}\right].
\]

For the area \( \mathcal{A} \) we can use the formula valid in any dimension:

\[
4\mathcal{A}^2 = |\mathbf{AB}|^2|\mathbf{AC}|^2 - (\mathbf{AB} \cdot \mathbf{AC})^2.
\]
Here we have:

\[
\begin{align*}
\mathbf{AB} &= \mathbf{B} - \mathbf{A} = [3, -1] - [4, 2] = [-1, -3], \\
|\mathbf{AB}|^2 &= (-1)^2 + (-3)^2 = 1 + 9 = 10, \\
\mathbf{AC} &= \mathbf{C} - \mathbf{A} = [5, -3] - [4, 2] = [1, -5], \\
|\mathbf{AC}|^2 &= 1^2 + (-5)^2 = 1 + 25 = 26, \\
\mathbf{AB} \cdot \mathbf{AC} &= [-1, -3] \cdot [1, -5] = -1 + 15 = 14.
\end{align*}
\]

So we get:

\[
\begin{align*}
4\mathcal{A}^2 &= |\mathbf{AB}|^2 |\mathbf{AC}|^2 - (\mathbf{AB} \cdot \mathbf{AC})^2 \\
&= (10)(26) - 196 = 260 - 196 = 64, \\
\mathcal{A} &= \sqrt{16} = 4.
\end{align*}
\]

Note that in two-dimensions we can use the formula (coming from the cross-product formula in three dimensions) that if the vectors along two sides of the triangle are the vectors \([a, b]\) and \([c, d]\) then the triangle area is given by:

\[
2\mathcal{A} = \left| \begin{array}{cc} a & b \\ c & d \end{array} \right| = |ad - bc|.
\]

Here we have the vectors \(\mathbf{AB} = [-1, -3]\) and \(\mathbf{AC} = [1, -5]\), found above, so we get:

\[
2\mathcal{A} = \left| \begin{array}{cc} -1 & -3 \\ 1 & -5 \end{array} \right| = |(-1)(-5) - (-3)(1)| = |5 + 3| = 8.
\]

So \(\mathcal{A} = 4\), as before.

So \(\mathcal{M} = \mathbf{AC} = 4 \left[ 4, -\frac{2}{3} \right] = \left[ 16, -\frac{8}{3} \right]\).

So the required flux is \([3, 0] \cdot \mathcal{M} = [3, 0] \cdot \left[ 16, -\frac{8}{3} \right] = 3(16) + 0 = 48\).
Find the work done by $\mathbf{F}$ in going once clockwise around the triangle $ABC$.

By the curl theorem the work done by $\mathbf{F}$ going once counter-clockwise around the triangle is (with $\mathbf{T}$ the unit tangent along the curve $s$ the parametrization by length along the curve):

$$\int_{\partial \mathcal{R}} \mathbf{F} \cdot \mathbf{T} ds = \int \int_{\mathcal{R}} \text{curl}(\mathbf{F}) dA.$$

We have:

$$\text{curl} \mathbf{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial(xy + 2x^2)}{\partial x} - \frac{\partial(x^2 - y^2)}{\partial y} = y + 4x - (-2y) = 4x + 3y,$$

$$\int_{\partial \mathcal{R}} \mathbf{F} \cdot \mathbf{T} ds = \int \int_{\mathcal{R}} \text{curl}(\mathbf{F}) dA = \int \int_{\mathcal{R}} 4x + 3y dA$$

$$= [4, 3] \int \int_{\mathcal{R}} [x, y] dA = [4, 3] \mathcal{M} = [4, 3] \left[ 16, -\frac{8}{3} \right] = 64 + 3 \left( -\frac{8}{3} \right) = 56.$$

For the given problem, we want the work done when going around the boundary of the triangle clockwise, not counter-clockwise, so the required work done is $-56$ units of work (Joules for the metric case).
Alternatively we can do these problems by doing the line integrals.

We have:

\[ \mathbf{F} \cdot \mathbf{N} ds = \mathbf{F} \cdot \left[ \frac{dy}{ds} - \frac{dx}{ds} \right] ds = [x^2 - y^2, xy + 2x^2].[dy, -dx] \]

\[ = (x^2 - y^2)dy - (xy + 2x^2)dx = d \left( x^2y - \frac{y^3}{3} \right) - 2xydx - (xy + 2x^2)dx \]

\[ = d \left( x^2y - \frac{y^3}{3} - \frac{2x^3}{3} \right) - 3xydx. \]

We can discard the exact differential, so we are left with integrating \( \alpha = -3xydx \) counter-clockwise around the triangle to obtain the flux.

Similarly we have:

\[ \mathbf{F} \cdot \mathbf{T} ds = \mathbf{F} \cdot \left[ \frac{dx}{ds}, \frac{dy}{ds} \right] ds = [x^2 - y^2, xy + 2x^2].[dx, dy] \]

\[ = (x^2 - y^2)dx + (xy + 2x^2)dy = d \left( \frac{xy^2}{2} + 2x^2y \right) - \left( \frac{1}{2}y^2 + 4xy \right) dx + (x^2 - y^2)dx \]

\[ = d \left( \frac{xy^2}{2} + 2x^2y + \frac{1}{3}x^3 \right) - \left( \frac{3}{2}y^2 + 4xy \right) dx. \]

We can discard the exact differential, so we are left with integrating \( -\beta = \left( \frac{3}{2}y^2 + 4xy \right) dx = y\left(3y + 8x\right) \) counter-clockwise around the triangle to obtain the required work done.
We go around the triangle from $B$ to $C$ to $A$ to $B$.

- The line $BC$ has slope $\frac{-3 - (-1)}{5 - 3} = \frac{-2}{2} = -1$, so has equation:
  \[ y = 2 - x. \]
Then we have:

\[
\alpha = -3xydx = -3x(2 - x)dx
\]

\[
= (-6x + 3x^2)dx = d(-3x^2 + x^3).
\]

So we get:

\[
\int_{BC} \alpha = \int_{BC} d(-3x^2 + x^3) = \left[-3x^2 + x^3\right]_{x=3}^{x=5}
\]

\[
= -3(5^2) + 5^3 - (-3(3^2) + 3^3) = -75 + 125 - 0 = 50.
\]

Also we have:

\[
-\beta = \frac{y}{2}(3y+8x)dx = \frac{1}{2}(2-x)(6+5x)dx = \left(6 + 2x - \frac{5}{2}x^2\right)dx = d\left(6x + x^2 - \frac{5}{6}x^3\right).
\]

So we get:

\[
\int_{BC} (-\beta) = \int_{BC} d\left(6x + x^2 - \frac{5}{6}x^3\right)
\]

\[
= \left[6x + x^2 - \frac{5}{6}x^3\right]_{x=3}^{x=5}
\]

\[
= 6(5 - 3) + 5^2 - 3^2 - \frac{5}{6}(5^3 - 3^3)
\]

\[
= 12 + 16 - \frac{5}{6}(98) = 28 - \frac{245}{3} = -16\frac{1}{3}.
\]
The line \( CA \) has slope \( \frac{-3 - 2}{5 - 4} = -5 \), so has equation: \( y = 22 - 5x \). Then we have:

\[
\alpha = -3xydx = -3x(22 - 5x)dx
= (-66x + 15x^2)dx = d(-33x^2 + 5x^3).
\]

So we get:

\[
\int_c \alpha = \int_c d(-33x^2 + 5x^3)
= [-33x^2 + 5x^3]_{x=4}^{x=5} = -33(4^2 - 5^2) + 5(4^3 - 5^3)
= -33(-9) + 5(-61) = 297 - 305 = -8.
\]

Also we have:

\[
-\beta = \frac{y}{2}(3y + 8x)dx = \frac{1}{2}(22 - 5x)(66 - 7x)dx
= \left(726 - 242x + \frac{35}{2}x^2\right)dx = d\left(726x - 121x^2 + \frac{35}{6}x^3\right).
\]

So we get:

\[
\int_c (-\beta) = \int_c d\left(726x - 121x^2 + \frac{35}{6}x^3\right)
= \left[726x - 121x^2 + \frac{35}{6}x^3\right]_{x=4}^{x=5}
= 726(-1) - 121(-9) + \frac{35}{6}(-61)
= -726 + 1089 - \frac{2135}{6} = \frac{1}{6}(2178 - 2135) = \frac{43}{6}.
\]
The line $AB$ has slope $\frac{-1 - 2}{3 - 4} = 3$, so has equation: $y = 3x - 10$. Then we have:

\[ \alpha = -3xydx = -3x(3x-10)dx = (30x - 9x^2)dx = d(15x^2 - 3x^3). \]

So we get:

\[ \int_{AB} \alpha = \int_{AB} d(15x^2 - 3x^3) = [15x^2 - 3x^3]_{x=4}^{x=3} = 15(3^2 - 4^2) - 3(3^3 - 4^3) = 15(-7) - 3(-37) = -105 + 111 = 6. \]

Also we have:

\[ -\beta = \frac{y}{2}(3y + 8x)dx = \frac{1}{2}(3x - 10)(17x - 30)dx = \left(150 - 130x + \frac{51}{2}x^2\right)dx = d\left(150x - 65x^2 + \frac{17}{2}x^3\right). \]

So we get:

\[ \int_{CA} (-\beta) = \int_{AB} d\left(150x - 65x^2 + \frac{17}{2}x^3\right) = \left[150x - 65x^2 + \frac{17}{2}x^3\right]_{x=4}^{x=3} = 150(-1) - 65(-7) + \frac{17}{2}(-37) = -150 + 455 - \frac{629}{2} = \frac{1}{2}(610 - 629) = -\frac{19}{2}. \]

Putting the pieces together, we get for the total flux:

\[ \int_{BC} \alpha + \int_{CA} \alpha + \int_{AB} \alpha = 50 - 8 + 6 = 48. \]

Finally for the work done, we get:

\[ \int_{BC} (-\beta) + \int_{CA} (-\beta) + \int_{AB} (-\beta) = \frac{1}{6}(-322 + 43 - 57) = -\frac{336}{6} = -56. \]

These results agree with the result obtained by using Green’s Theorem, as above.
Question 2

Let \( F = [x^2 - yz, y^2 - zx, z^2 - xy] \).

Let \( R \) be the region in space bounded by the sphere \( S \) centered at \((3, 0, 0)\) of radius 3 units.

- Sketch \( S \).
- Find the work done by the force \( F \) in going from \( A = (3, 3, 0) \) to the origin along the short arc of the great circle of the sphere \( S \) connecting \( A \) to the origin.

We have:

\[
\alpha = \overrightarrow{F} \cdot d\overrightarrow{X} \\
= [x^2 - yz, y^2 - zx, z^2 - xy]. [dx, dy, dz] \\
= x^2 dx + y^2 dy + z^2 dz - (yzdx + zxdy + xydz) \\
= \frac{1}{3} d(x^3 + y^3 + z^3 - 3xyz).
\]

So the required work done in units of work is:

\[
\int \alpha = \int_{(3,3,0)}^{(0,0,0)} \frac{1}{3} d(x^3 + y^3 + z^3 - 3xyz) \\
= \frac{1}{3} [x^3 + y^3 + z^3 - 3xyz]_{(3,3,0)}^{(0,0,0)} \\
= 0 - \frac{1}{3}(3^3 + 3^3 + 0 - 0) = -18.
\]
Alternatively we parametrize the circle:
\[ \vec{X} = [x, y, z] = [3 + 3 \cos(t), 3 \sin(t), 0], \]
\[ d\vec{X} = 3[- \sin(t), \cos(t), 0] dt, \]
\[ \vec{F} = [x^2 - yz, y^2 - zx, z^2 - xy] = [9(1 + \cos(t))^2, 9 \sin^2(t), -9 \sin(t) \cos(t)], \]
\[ \alpha = \vec{F} \cdot d\vec{X} = 27[(1 + \cos(t))^2, \sin^2(t), -\sin(t) \cos(t)] [-\sin(t), \cos(t), 0] dt \]
\[ = 27(\sin^2(t) \cos(t) - \sin(t)(1 + \cos(t))^2) dt \]
The \( t \)-range is from \( t = \frac{\pi}{2} \) to \( \pi \).
So the required integral is:
\[ \int_{\pi/2}^{\pi} (27(\sin^2(t) \cos(t) - \sin(t)(1 + \cos(t))^2)) dt \]
\[ = 9[\sin^3(t) + (1 + \cos(t))^3]_{\pi/2}^{\pi} = 9(0 - (1 + 1)) = -18. \]

- Find the flux of \( \vec{F} \) outside the region \( \mathcal{R} \).

By the divergence theorem, we need the integral over the interior of the sphere of the divergence of \( \vec{F} \).

But the divergence of \( \vec{F} \) is:
\[ \text{div}(\vec{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial(x^2 - yz)}{\partial x} + \frac{\partial(y^2 - zx)}{\partial y} + \frac{\partial(z^2 - xy)}{\partial z} \]
\[ = 2x + 2y + 2z. \]
So the flux is (if \( \mathcal{R} \) denotes the interior of the sphere):
\[ \int_{\mathcal{R}} (2x + 2y + 2z) dV = 2[1, 1, 1]. \int_{\mathcal{R}} [x, y, z])dV \]
\[ = 2[1, 1, 1].\mathcal{M} = 2[1, 1, 1].\mathcal{C}V. \]
Here \( \mathcal{M} \) is the total vector moment of the sphere about the origin, \( \mathcal{V} \) is its volume and \( \mathcal{C} \) is its centroid.
So here \( \mathcal{V} = \frac{4\pi}{3}(3^3) = 108\pi \) and \( \mathcal{C} = [3, 0, 0] \), so we get for the required flux:
\[ 2[1, 1, 1].[3, 0, 0](108\pi) = 648\pi = 2035.752. \]
Question 3

Solve each of the following differential equations and discuss the behavior of each solution as a function of the variable $t$:

- $y'' - 2y' - 3y = 5e^{-2t}, \quad y(0) = 4, \quad y'(0) = 2$

For a particular solution, we try:

$$y = Ae^{-2t},$$
$$y' = Ae^{-2t}(-2),$$
$$y'' = Ae^{-2t}(4),$$

$$0 = y'' - 2y' - 3y - 5e^{-2t} = e^{-2t}(-5 + 4A - 2(-2A) - 3A) = e^{-2t}(-5 + 5A).$$

So $A = 1$ gives the particular solution:

$$y_p = e^{-2t}.$$ 

The associated homogeneous equation is:

$$y'' - 2y' - 3y = 0.$$ 

We try an exponential:

$$y = e^{mt},$$
$$y' = e^{mt}(m),$$
$$y'' = e^{mt}(m^2),$$

$$0 = y'' - 2y' - 3y = e^{mt}(m^2 - 2m - 3) = e^{mt}(m - 3)(m + 1).$$

So $m = 3$, or $m = -1$ and the general solution of the associated homogeneous equation is:

$$y_h = Be^{3t} + Ce^{-t}.$$
So the general solution of the given differential equation is:

\[ y = y_p + y_h = e^{-2t} + Be^{3t} + Ce^{-t}. \]

Then we have, for its derivative:

\[ y' = -2e^{-2t} + 3Be^{3t} - Ce^{-t}. \]

We put \( t = 0 \) in these last two formulas, and use the initial conditions, \( y(0) = 4 \) and \( y'(0) = 2 \), giving:

\[
\begin{align*}
y(0) &= 1 + B + C = 4, \quad B + C = 3, \\
y'(0) &= -2 + 3B - C = 2, \quad 3B - C = 4.
\end{align*}
\]

Adding these two equations gives \( 4B = 7 \), so \( B = \frac{7}{4} \).

Then we get:

\[ C = 3 - B = 3 - \frac{7}{4} = \frac{5}{4}. \]

So the required solution is:

\[ y = \frac{1}{4}(4e^{-2t} + 7e^{3t} + 5e^{-t}). \]

This blows up to infinity in both directions in time, with the only critical point occurring at the only zero of \( y' \) which gives the global minimum for \( y \) at \((-0.1036, 3.899)\).

Note that \( y'' = \frac{1}{4}(16e^{-2t} + 63e^{3t} + 5e^{-t}) > 0 \), so the graph is defined for all time and is everywhere concave up and is increasing for \( t \geq -0.1036 \) and decreasing for \( t \leq -0.1036 \).
\[ y'' + 9y = 2\cos(2t), \quad y(0) = 0, \quad y'(0) = 3 \]

For a particular solution, we try:

\[
y = A \cos(2t) + B \sin(2t),
\]

\[
y' = -2A \sin(2t) + 2B \cos(2t),
\]

\[
y'' = -4A \cos(2t) - 4B \sin(2t),
\]

\[
0 = y'' + 9y - 2 \cos(2t) = \cos(2t)(-4A - 4B + 9A) + \sin(2t)(-4B + 9B)
\]

\[
= \cos(2t)(-4A + 6B) + \sin(2t)(5B).
\]

So \( A = \frac{2}{5}, \ B = 0 \) gives the particular solution:

\[
y_p = \frac{2}{5} \cos(2t).
\]

The associated homogeneous equation is:

\[
y'' + 9y = 0.
\]

We try an exponential:

\[
y = e^{mt},
\]

\[
y' = e^{mt}(m),
\]

\[
y'' = e^{mt}(m^2),
\]

\[
0 = y'' + 9y = e^{mt}(m^2 + 9) = e^{mt}(m - 3i)(m + 3i).
\]

So \( m = \pm 3i \) and the general solution of the associated homogeneous equation is:

\[
y_h = Ce^{3it} + De^{-3it} = C(\cos(3t) + i \sin(3t)) + D(\cos(3t) - i \sin(3t))
\]

\[
= P \cos(3t) + Q \sin(3t).
\]
So the general solution of the given differential equation is:

\[ y = y_p + y_h = \frac{2}{5} \cos(2t) + P \cos(3t) + Q \sin(3t). \]

Then we have, for its derivative:

\[ y' = -\frac{4}{5} \sin(2t) - 3P \sin(3t) + 3Q \cos(3t). \]

We put \( t = 0 \) in these last two formulas, and use the initial conditions, \( y(0) = 0 \) and \( y'(0) = 3 \), giving:

\[ y(0) = \frac{2}{5} + P = 0, \quad P = -\frac{2}{5}, \]

\[ y'(0) = 3Q = 3, \quad Q = 1. \]

So the required solution is:

\[ y = \frac{1}{5}(2 \cos(2t) - 2 \cos(3t) + 5 \sin(3t)). \]
Plotting gives a periodic motion with period $2\pi$.
Each cycle has three local minima (one of these is a global minimum)
and three local maxima (one of these is a global maximum).

- Starting the cycle at $t = 0$, the curve is initially concave up and
  increasing.
- It switches to concave down at the first inflection point [0.07214122831, 0.2199131842]
- It rises to the first local maximum at [0.5744821182, 1.212967793]
  and then decreases.
- It switches to concave up at the second inflection point [1.138277346, −0.1441358271]
- It decreases to the first local minimum at [1.679776757, −1.466024378]
  and then increases.
- It switches to concave down at the third inflection point [2.234494312, −0.053564759].
- It rises to the second local maximum at [2.797470282, 1.372575298]
  and then decreases.
- It switches to concave up at the fourth inflection point [3.320184048, 0.2081967891]
- It decreases to the second local minimum at [3.874964893, −1.002320224]
  and then increases.
- It switches to concave down at the fifth inflection point [4.357466738, −0.1685473842].
- It rises to the third local maximum at [4.863869974, 0.6923076475]
  and then decreases.
- It switches to concave up at the sixth inflection point [5.346412350, −0.0662546371]
- It decreases to the third local minimum at [5.820004651, −0.8154250165]
  and then increases.

In the first period, the $t$-intercepts are at:

$$t = 0, 1.100485879, 2.247870049, 3.380461756, 4.420674065 \text{ and } 5.319484273.$$  

The range of the solution is $[-1.466024378, 1.372575298]$. 

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