Calculus III Quiz 8 Solutions 11/18/5

We begin with the following integrals:

- The integral \( \int \cos^2(\theta) d\theta \):
  \[
  \int \cos^2(\theta) d\theta = \frac{1}{2} \int (1 + \cos(2\theta)) d\theta \\
  = \frac{1}{2} (\theta + \frac{1}{2} \sin(2\theta)) + C = \frac{1}{2} (\theta + \sin(\theta) \cos(\theta)) + C,
  \]

- The integral \( \int \sin^2(\theta) d\theta \):
  \[
  \int \sin^2(\theta) d\theta = \frac{1}{2} \int \frac{1}{2} (1 - \cos(2\theta)) d\theta \\
  = \frac{1}{2} (\theta - \frac{1}{2} \sin(2\theta)) + C = \frac{1}{2} (\theta - \sin(\theta) \cos(\theta)) + C,
  \]

- The integral \( \int \cos^3(\theta) d\theta \):
  \[
  \int \cos^3(\theta) d\theta = \int \cos(\theta)(1 - \sin^2(\theta)) d\theta \\
  = \int (1 - u^2) du = u - \frac{u^3}{3} + C = \frac{1}{3} (3 \sin(\theta) - \sin^3(\theta)) + C.
  \]
  Here we put \( u = \sin(t) \), \( du = \cos(t) dt \) and \( 1 - u^2 = 1 - \sin^2(t) = \cos^2(t) \).

- The integral \( \int \cos^4(\theta) d\theta \):
  \[
  \int \cos^4(\theta) d\theta = \frac{1}{4} \int (1 + \cos(2\theta))^2 d\theta = \frac{1}{4} \int (1 + 2 \cos(2\theta) + \cos^2(2\theta)) d\theta \\
  = \frac{1}{8} \int (2 + 4 \cos(2\theta) + 1 + \cos(4\theta)) d\theta \\
  = \frac{1}{8} \left( 3\theta + 2 \sin(2\theta) + \frac{1}{4} \sin(4\theta) \right) + C = \frac{1}{8} \left( 3\theta + 4 \sin(\theta) \cos(\theta) + \frac{1}{2} \sin(2\theta) \cos(2\theta) \right) + C \\
  = \frac{1}{8} (3\theta + 3 \sin(\theta) \cos(\theta) + \sin(\theta) \cos(\theta)(2 \cos^2(\theta) - 1)) + C \\
  = \frac{1}{8} (3\theta + 3 \sin(\theta) \cos(\theta) + 2 \sin(\theta) \cos^3(\theta)) + C.
  \]
Question 1

Sketch the region between the two circles with the polar equations \( r = 2 \cos(\theta) \) and \( r = 4 \cos(\theta) \) and determine its centroid.

The circle \( r = 4 \cos(\theta) \) has center at \((2, 0)\) and radius 2 units. The circle \( r = 2 \cos(\theta) \) has center at \((1, 0)\) and radius 1 units. Both circles pass through the origin. Each is traced once around in a counter-clockwise direction, starting from the origin, by taking the \( \theta \) range \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \).

We can, if we wish, obtain their Cartesian equations as follows.

For a a given constant, multiply the equation \( r = 2a \cos(\theta) \) by \( r \) and use the relations \( r^2 = x^2 + y^2 \) and \( r \cos(\theta) = x \), giving:

\[
\begin{align*}
    r^2 &= 2ar \cos(\theta), \quad x^2 + y^2 = 2ax, \\
    x^2 - 2ax + y^2 &= 0, \quad (x - a)^2 - a^2 + y^2 = 0, \\
    (x - a)^2 + y^2 &= a^2.
\end{align*}
\]

We see that we have the standard equation of a circle of radius \( a \) centered at the point \((a, 0)\).

For the given problem we have the special cases \( a = 1 \) for the inner circle and \( a = 2 \) for the outer circle, as described above.

By symmetry, the required centroid is at \((X, 0)\), where:

\[
X = \frac{\int \int x dA}{\int \int dA} = \frac{\int \int r \cos(\theta) r dr d\theta}{\int \int r dr d\theta} = \frac{1}{\mathcal{A}} \int \int r^2 \cos(\theta) dr d\theta.
\]

The area \( \mathcal{A} \) of the given region is:

\[
\mathcal{A} = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{1 \cos(\theta)} r dr d\theta = \frac{1}{2} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \left[r^2\right]_{\frac{1 \cos(\theta)}{2 \cos(\theta)}}^{1 \cos(\theta)} d\theta
= \frac{1}{2} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} ((4 \cos(\theta))^2 - (2 \cos(\theta))^2) d\theta
= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (16 \cos^2(\theta) - 4 \cos^2(\theta)) d\theta
= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (6 \cos^2(\theta)) d\theta = 3[\theta + \sin(\theta) \cos(\theta)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}
= 3 \left(\frac{\pi}{2} + 0 - \left(-\frac{\pi}{2} + 0\right)\right) = 3\pi.
\]
The \( x \)-moment is:

\[
M_x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{2\cos(\theta)}^{4\cos(\theta)} r^2 \cos(\theta) dr d\theta = \frac{1}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta) \left( r^3 \right)_{2\cos(\theta)} d\theta
\]

\[
= \frac{1}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta) \left( (4 \cos(\theta))^3 - (2 \cos(\theta))^3 \right) d\theta = \frac{1}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta) (64 \cos^3(\theta) - 8 \cos^3(\theta)) d\theta
\]

\[
= \frac{56}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4(\theta) d\theta = \frac{56}{3} \left( \frac{1}{8} \right) \left[ 3\theta + 3 \sin(\theta) \cos(\theta) + 2 \sin(\theta) \cos^3(\theta) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}
\]

\[
= \frac{7}{3} \left( \frac{3\pi}{2} + 0 - \left( -3\frac{\pi}{2} + 0 \right) \right) = \frac{7\pi}{3}.
\]

So \( X = \frac{M_x}{A} = \frac{7\pi}{3\pi} = \frac{7}{3} \) and the centroid of the given region is at the point \( \left[ \frac{7}{3}, 0 \right] = [2.3, 0] \).

Alternatively, we can do this problem without calculus, as follows.

The given region is obtained by subtracting a smaller circular disc from a larger disc.

- The larger disc has radius 2 and centroid at its center \( C_1 = [2, 0] \), so has area \( A_1 = \pi(2^2) = 4\pi \) and total moment vector:

\[
M_1 = A_1 C_1 = 4\pi [2, 0] = [8\pi, 0].
\]

- The smaller disc has its center \( C_2 = [1, 0] \), so has area \( A_2 = \pi(1^2) = \pi \) and total moment vector:

\[
M_2 = A_2 C_2 = \pi [1, 0] = [\pi, 0].
\]

By the addition principle for moments, the moment vector \( \overline{M} \) of the given region is then:

\[
\overline{M} = M_1 - M_2 = [8\pi, 0] - [\pi, 0] = 7\pi.
\]

The total area \( A \) of the given region is:

\[
A = A_1 - A_2 = 4\pi - \pi = 3\pi.
\]

So the required centroid \( \overline{C} \) is given by:

\[
\overline{C} = \frac{\overline{M}}{\overline{A}} = \frac{1}{3\pi} [7\pi, 0] = \left[ \frac{7}{3}, 0 \right] = [2.3, 0].
\]
Question 2

- Sketch the ellipse with equation \( \frac{x^2}{9} + \frac{y^2}{25} = 1 \).

This is a standard ellipse centered at the origin, symmetric about the axes, with major axis along the y-axis from \((-5, 0)\) to \((5, 0)\) and with minor axis along the x-axis from \((-3, 0)\) to \((3, 0)\).

- Parametrize the ellipse.

Using the trigonometric identity, \( \cos^2(t) + \sin^2(t) = 1 \), we write:

\[
(x, y) = (3 \cos(t), 5 \sin(t)).
\]

Here the parameter \( t \) goes from 0 to \( 2\pi \) to cover the ellipse once counter-clockwise. Then we have, for the differentials:

\[
dx = -3 \sin(t) \, dt, \quad dy = 5 \cos(t) \, dt.
\]

- Find the line integral \( \int_{\Gamma} (x \, dx - y \, dy) \), where \( \Gamma \) is the ellipse, traced once counter-clockwise.

We have, using the above parametrization:

\[
\alpha = x \, dx - y \, dy = 3 \cos(t)(-3 \sin(t) \, dt) - 5 \sin(t)(5 \cos(t) \, dt) = -34 \sin(t) \cos(t) \, dt = -17 \sin(2t) \, dt,
\]

\[
\int_{\Gamma} (x \, dx - y \, dy) = \int_{\Gamma} \alpha = \int_{0}^{2\pi} (-17 \sin(2t)) \, dt = \frac{17}{2} \sin(2t) \bigg|_{0}^{2\pi} = \frac{17}{2} [1-1] = 0.
\]

Alternatively, we see that \( x \, dx - y \, dy = d \left( \frac{1}{2} (x^2 - y^2) \right) \) is exact. So by the generalized Fundamental Theorem of Calculus, its integral around any closed curve is zero.

- Find the line integral \( \int_{\Gamma} (x \, dy - y \, dx) \), where \( \Gamma \) is the ellipse, traced once counter-clockwise.

We have, using the above parametrization:

\[
\beta = x \, dy - y \, dx = 3 \cos(t)(5 \cos(t) \, dt) - 5 \sin(t)(-3 \sin(t) \, dt) = 15(\cos^2(t) + \sin^2(t)) \, dt = 15 \, dt,
\]

\[
\int_{\Gamma} (x \, dy - y \, dx) = \int_{\Gamma} \beta = \int_{0}^{2\pi} 15 \, dt = 15[t]_{0}^{2\pi} = 15(2\pi) = 30\pi.
\]
Question 3

Find the line integral of \( \alpha = xdy - ydx \) taken over the triangle \( ABC \) in the plane with vertices \( A = [0, 0] \), \( B = [2, 0] \) and \( C = [1, 1] \), traced once counter-clockwise.

We parametrize the sides of the triangle, compute the differential and then integrate:

- \( \Gamma_1 \), the side \( AB \): \( x = x, y = 0, dx = dx, dy = 0 \), the parameter \( x \) ranges from 0 to 2.
  Then we have:
  \[
  \alpha = xdy - ydx = 0 - 0 = 0,
  \]
  \[
  \int_{\Gamma_1} \alpha = 0.
  \]

- \( \Gamma_2 \), the side \( BC \): \( x = x, y = 2 - x, dx = dx, dy = -dx \), the parameter \( x \) ranges from 2 to 1.
  Then we have:
  \[
  \alpha = xdy - ydx = -xdx - (2 - x)dx = -2dx,
  \]
  \[
  \int_{\Gamma_2} \alpha = \int_{2}^{1} -2dx = -[2x]_{2}^{1} = -2 - (-4) = 2.
  \]

- \( \Gamma_3 \), the side \( CA \): \( x = x, y = x, dx = dx, dy = dx \), the parameter \( x \) ranges from 1 to 0.
  Then we have:
  \[
  \alpha = xdy - ydx = xdx - xdx = 0,
  \]
  \[
  \int_{\Gamma_3} \alpha = 0.
  \]

Then the required integral around the triangle \( ABC \) is:
\[
\int_{\Gamma_1} \alpha + \int_{\Gamma_2} \alpha + \int_{\Gamma_3} \alpha = 0 + 2 + 0 = 2.
\]
Alternatively if $\Gamma$ is the triangle $ABC$ traced once counter-clockwise, and $\Delta$ is the inside of the triangle $ABC$, Green’s Theorem says that:

$$\int_{\Gamma} (P \, dx + Q \, dy) = \int \int_{\Delta} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA.$$

Here we have $P \, dx + Q \, dy = -y \, dx + x \, dy$, so we have:

- $Q = x$, so $\frac{\partial Q}{\partial x} = 1$.

- $P = -y$, so $\frac{\partial P}{\partial y} = -1$.

- So we get $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2$.

Then Green’s Theorem gives:

$$\int_{\Gamma} (x \, dy - y \, dx) = \int \int_{\Delta} 2 \, dA = 2 \mathcal{A}.$$

Here $\mathcal{A}$ is the area of the triangle $ABC$.

But the triangle $ABC$ has base 2 units and height 1 unit, so its area is $\mathcal{A} = \frac{1}{2} (2)(1) = 1$. So we get:

$$\int_{\Gamma} (x \, dy - y \, dx) = \int \int_{\Delta} 2 \, dA = 2 \mathcal{A} = 2(1) = 2.$$