Calculus III Quiz 4 Solutions 9/23/5

Question 1
Let \( f(x, y) = x^2 + xy - 2y^2 - 8x + 5y + 7. \)

- Find the equation of the tangent plane to the surface \( z = f(x, y) \) at the point \((2, 3, -2)\).

We have, evaluating at \( x = 2 \) and \( y = 3 \):

\[
\frac{\partial z}{\partial x} = 2x + y - 8 = 2(2) + 3 - 8 = 4 + 3 - 8 = -1,
\]

\[
\frac{\partial z}{\partial y} = x - 4y + 5 = 2 - 4(3) + 5 = 2 - 12 + 5 = -5.
\]

So vectors in the tangent plane are:

\[
V = [1, 0, \frac{\partial z}{\partial x}] = [1, 0, -1],
\]

\[
W = [0, 1, \frac{\partial z}{\partial y}] = [0, 1, -5].
\]

The normal to the tangent plane is then:

\[
N = V \times W
\]

\[
= \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 0 & -1 \\
0 & 1 & -5
\end{vmatrix}
= \hat{i} \begin{vmatrix} 0 & -1 \\ 1 & -5 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & -1 \\ 0 & -5 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}
= \hat{i}(0 + 1) - \hat{j}(-5 - 0) + \hat{k}(1 - 0) = \hat{i} + 5\hat{j} + \hat{k}.
\]

The plane goes through the point \((2, 3, -2)\) and has normal \((1, 5, 1)\), so it has equation:

\[
1(x - 2) + 5(y - 3) + (z - (-2)) = 0,
\]

\[
x + 5y + z = 15.
\]
Find all points where the tangent plane to the surface \( z = f(x, y) \) is horizontal.

We need both partial derivatives of \( z \) to vanish:

\[
0 = \frac{\partial z}{\partial x} = 2x + y - 8,
\]
\[
0 = \frac{\partial z}{\partial y} = x - 4y + 5.
\]

Multiplying the first equation by 4 gives \( 8x + 4y - 32 = 0 \).

Adding this to the second equation gives:

\[
0 = 8x + 4y - 32 + x - 4y + 5 = 9x - 27,
\]

so \( x = 3 \).

When \( x = 3 \), we have \( y = 8 - 2x = 8 - 2(3) = 8 - 6 = 2 \).

So the tangent plane is horizontal when \( x = 3 \) and \( y = 2 \).

The corresponding \( z \)-value is \( 3^2 + 3(2) - 2(2)^2 - 8(3) + 5(2) + 7 = 9 + 6 - 8 - 24 + 10 + 7 = 0 \).

So the tangent plane at \( (3, 2, 0) \) is the only one that is horizontal.

Show that the level curve \( f(x, y) = 0 \) is a pair of lines (factorize the function \( f(x, y) \)) and sketch these lines.

We have: \( x^2 + xy - 2y^2 = (x + 2y)(x - y) \), so we try to factor \( f \) as follows:

\[
x^2 + xy - 2y^2 - 8x + 5y + 7 = (x + 2y + a)(x - y + b)
\]
\[
= x^2 + xy - 2y^2 + x(a + b) + y(2b - a) + ab.
\]

So we need \( a + b = -8 \), \( 2b - a = 5 \) and \( ab = 7 \).

Adding the first two equations gives \( 3b = -3 \), so \( b = -1 \).

Then \( a = -7 \) and all three equations are satisfied, giving:

\[
f(x, y) = x^2 + xy - 2y^2 - 8x + 5y + 7 = (x + 2y - 7)(x - y - 1).
\]

So \( f(x, y) = 0 \) gives the two lines \( x + 2y - 7 = 0 \) and \( x - y - 1 = 0 \).

These meet where:

\[
y = x - 1, \quad 0 = x + 2(x - 1) - 7 = 3x - 9,
\]
\[
x = 3, \quad y = 3 - 1 = 2.
\]

So the two lines cross at \( (3, 2) \), one has slope 1 and the other slope \(-\frac{1}{2}\).
• Explain why the other level curves of $f(x, y)$ are hyperbolas.

The curve $z = c$ is:

$$(x + 2y - 7)(x - y - 1) = c.$$ 

If $c = 0$, this gives a pair of lines as described above.
If $c \neq 0$, this gives a hyperbola with asymptotes the same pair of lines.

In general the curve $ax^2 + bxy + cy^2 + px + qy + r = 0$ is:

- A hyperbola or line pair iff $b^2 > 4ac$ iff $ax^2 + bxy + cy^2$ factors with real factors.
  The special case of the line pair occurs only if the equation factors completely.
- A parabola or repeated line iff it has at least one point and $b^2 = 4ac$ iff it has at least one point and $ax^2 + bxy + cy^2$ is a perfect square.
  The special case of the repeated line occurs only if the equation is a perfect square.
- An ellipse or circle iff it has at least two points and $b^2 < 4ac$, iff it has at least two points and $ax^2 + bxy + cy^2$ has no real factors.
  The special case of a circle requires that $b = 0$ and $a = c \neq 0$.

Here $a = b = 1$ and $c = -2$, so $b^2 - 4ac = 1 - 4(1)(-2) = 9 > 0$, so the curve is either a line pair or a hyperbola.
The line pair is the case when the equation factors completely, so only when $c = 0$. 
Question 2
Let $\mathbf{A} = [4, -3, -4]$, $\mathbf{B} = [1, 3, 8]$ and $\mathbf{C} = [-1, 1, 6]$ and $D = [2, 0, 5]$.

- Find the equation of the plane $ABC$.

We have:

$$\mathbf{V} = \mathbf{AB} = \mathbf{B} - \mathbf{A} = [1, 3, 8] - [4, -3, -4] = [-3, 6, 12],$$
$$\mathbf{W} = \mathbf{AC} = \mathbf{C} - \mathbf{A} = [-1, 1, 6] - [4, -3, -4] = [-5, 4, 10].$$

So the plane has normal:

$$\mathbf{N} = \mathbf{V} \times \mathbf{W}$$

$$= \det \begin{vmatrix} i & j & k \\ -3 & 6 & 12 \\ -5 & 4 & 10 \end{vmatrix} = i \det \begin{vmatrix} 6 & 12 \\ 4 & 10 \end{vmatrix} - j \det \begin{vmatrix} -3 & 12 \\ -5 & 10 \end{vmatrix} + k \det \begin{vmatrix} 3 & 12 \\ -5 & 4 \end{vmatrix}$$

$$= i(60 - 48) - j(-30 + 60) + k(-12 + 30) = 12i - 30j + 18k.$$

The plane goes through the point $(4, -3, -4)$ and has normal $(12, -30, 18)$, so it has equation:

$$12(x - 4) - 30(y + 3) + 18(z - (-4)) = 0,$$
$$2(x - 4) - 5(y + 3) + 3(z + 4) = 0,$$
$$2x - 5y + 3z - 11 = 0.$$

- Find the parametric equations of the line through $D$ perpendicular to the plane $ABC$.

The direction of the required line is parallel to the normal to the plane $ABC$, so (after scaling the normal by a factor of $\frac{1}{6}$) may be taken to be the vector: $[2, -5, 3]$.

So the required parametric equations are:

$$\mathbf{X} = [2, 0, 5] + t[2, -5, 3],$$

$$x = 2 + 2t, \quad y = -5t, \quad z = 5 + 3t.$$
• Find the distance of the point $D$ from the plane $ABC$.

If the plane is $\frac{N \cdot X}{|N|} = c$, then the required distance of $\overline{D}$ from the plane is $\frac{|N \cdot D - c|}{|N|}$.

So here we take $N = [2, -5, 3]$, $D = [2, 0, 5]$ and $c = 11$. Then the required distance is:

$$\frac{|[2, -5, 3].[2, 0, 5] - 11|}{|[2, -5, 3]|} = \frac{|4 + 0 + 15 - 11|}{\sqrt{2^2 + (-5)^2 + 3^2}}$$

$$= \frac{8}{\sqrt{4 + 25 + 9}} = \frac{8}{\sqrt{38}} = 1.29777.$$

• Find the mirror image of the point $D$ in the plane $ABC$.

Let $Q$ be the point on the plane $ABC$ nearest to $D$.
This point is where the perpendicular line from $D$ to the plane meets the plane, so where the parametric equation of the line obeys the equation of the plane:

$$0 = 2(2+2t) - 5(-5t) + 3(5+3t) - 11 = 4 + 4t + 25t + 15 + 9t - 11 = 38t + 8,$$

$$t = -\frac{8}{38} = -\frac{4}{19}.$$

So the point $Q$ has $t = -\frac{4}{19}$.
So the mirror image of $D$ has $t = -\frac{8}{19}$, so is the point:

$$(x, y, z) = 2 + 2 \left( -\frac{8}{19} \right), \quad y = -5 \left( -\frac{8}{19} \right), \quad z = 5 + 3 \left( -\frac{8}{19} \right)$$

$$= \frac{1}{19} (38 - 16, 40, 95 - 24) = \frac{1}{19} (22, 40, 71) = [1.15789, 2.10526, 3.73684].$$
• Find the volume of the tetrahedron through the points $A$, $B$, $C$ and $D$.

We first the volume of the parallelopiped determined by the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$:

- $\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = [1, 3, 8] - [4, -3, -4] = [-3, 6, 12] = [-3, 6, 12]$, 
- $\overrightarrow{AC} = \overrightarrow{C} - \overrightarrow{A} = [-1, 1, 6] - [4, -3, -4] = [-5, 4, 10]$, 
- $\overrightarrow{AD} = \overrightarrow{D} - \overrightarrow{A} = [2, 0, 5] - [4, -3, -4] = [-2, 3, 9]$.

Then the parallelopiped volume is:

$$\begin{vmatrix}
-3 & 6 & 12 \\
-5 & 4 & 10 \\
-2 & 3 & 9 \\
\end{vmatrix}
= -3 \begin{vmatrix}
4 & 10 \\
-2 & 9 \\
\end{vmatrix}
- 6 \begin{vmatrix}
5 & 10 \\
-2 & 9 \\
\end{vmatrix}
+ 12 \begin{vmatrix}
5 & 4 \\
-2 & 3 \\
\end{vmatrix}
= -3(36 - 30) - 6(-45 + 20) + 12(-15 + 8)
= -3(6) - 6(-25) + 12(-7) = -18 + 150 - 84 = 48.$$

Alternatively the same volume is:

$$(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} = (12, -30, 18) \cdot (-2, 3, 9)
= 12(-2) - 30(3) + 18(9) = -24 - 90 + 162 = 48.$$

Then the tetrahedron volume is one sixth of the corresponding parallelopiped volume, so is $\frac{48}{6} = 8$ units of volume.