Calculus III  Homework 7 Solutions, due 10/19/5

Question 1

Consider the function, $G$, defined for $(x, y) \in \mathbb{R}^2$ by the formulas:

$$G(x, y) = \frac{x^3 + y^3}{x^2 + y^2}, \quad \text{if } (x, y) \neq (0, 0); \quad G(0, 0) = 0.$$

- Show that $G(x, y)$ is everywhere continuous.

Away from the origin, the function $G$ is a quotient of polynomials, and the denominator is non-zero, so $G$ is continuous.

At the origin, we need to show that the following limit is valid:

$$\lim_{(x, y) \to (0, 0)} \frac{x^3 + y^3}{x^2 + y^2} = G(0, 0) = 0.$$

Put $x = r \cos(\theta)$ and $y = r \sin(\theta)$, with $r > 0$.

Then $x^2 + y^2 = r^2$ and the limit becomes:

$$\lim_{(x, y) \to (0, 0)} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{r \to 0^+} \frac{r^3(\cos^3(\theta) + \sin^3(\theta))}{r^2} = \lim_{r \to 0^+} r(\cos^3(\theta) + \sin^3(\theta)) = 0,$$

since we have one factor $r$ going to zero and the other factor $\cos^3(\theta) + \sin^3(\theta)$ is no larger in size than 2.

So $G$ is continuous at the origin and so $G$ is everywhere continuous, as required.

- Show that $G$ has range all the reals.

Keep $y$ fixed at $y = 1$.

Then $G(x, 1) = \frac{x^3 + 1}{x^2 + 1}$.

This is defined for all $x$, is continuous and for large $x$ goes like $\frac{x^3}{x^2} = x$, so goes to $\infty$ as $x$ goes to $\infty$ and goes to $-\infty$ as $x$ goes to $-\infty$.

So $G(x, 1)$ takes all real values.

So $G(x, y)$ takes all real values.

So $G$ has range all the reals.
Show that $G(x, y)$ is differentiable everywhere except at the origin. Is $G$ differentiable at the origin? Explain your answer.

Away from the origin, we have:

$$\frac{\partial G}{\partial x} = \frac{3x^2(x^2 + y^2) - 2x(x^3 + y^3)}{(x^2 + y^2)^2},$$

$$\frac{\partial G}{\partial y} = \frac{3y^2(x^2 + y^2) - 2y(x^3 + y^3)}{(x^2 + y^2)^2}.$$

These partial derivatives are clearly continuous, so $G$ is differentiable away from the origin.

We have $G(x, 0) = x$, for all $x$, so $\frac{\partial G}{\partial x}|_{0,0} = 1$.

We have $G(0, y) = y$, for all $y$, so $\frac{\partial G}{\partial y}|_{0,0} = 1$.

If $G$ were differentiable at $(0, 0)$ then we would have $G'(0, 0) = [1, 1]$ and $G'((0, 0))(h, k) = [1, 1]$. so $[h, k] = h + k$.

So by the definition of the derivative, this requires that:

$$G(h, k) - G(0, 0) - (h + k) = \sqrt{h^2 + k^2}R(h, k),$$

where $R(h, k) \to 0$, as $(x, y) \to (0, 0)$.

But we have:

$$R(h, k) = \frac{1}{\sqrt{h^2 + k^2}} \left( \frac{h^3 + k^3}{h^2 + k^2} - (h + k) \right)$$

$$= \frac{1}{(\sqrt{h^2 + k^2})^3} \left( (h^2 - hk + k^2 - h^2 - k^2)(h + k) \right)$$

$$= -\frac{1}{(\sqrt{h^2 + k^2})^3} (hk)(h + k)$$

$$= -\sin(t) \cos(t)(\sin(t) + \cos(t)).$$

Here we put $h = r \cos(t)$ and $k = r \sin(t)$, with $r > 0$.

Clearly the limit as $r \to 0^+$ depends on $t$, so does not exist.

So $G$ is not differentiable at the origin.
Question 2

For $X = [x, y] \neq [0, 0]$, let a vector function $F(X)$ be given by:

$$F(X) = \frac{X}{|X|^2} = \frac{X}{X \cdot X} = \frac{1}{x^2 + y^2}[x, y].$$

- Compute the derivative matrix $F'(X)$ and show that its trace is zero, and that its determinant never vanishes on the domain of $F$.

We have:

$$F(X) = \begin{bmatrix} x \\ \frac{xy}{x^2 + y^2} \\ \frac{-yx}{x^2 + y^2} \end{bmatrix}$$

$$F'(X) = \nabla \left( \begin{bmatrix} x \\ \frac{xy}{x^2 + y^2} \\ \frac{-yx}{x^2 + y^2} \end{bmatrix} \right)$$

$$= (x^2 + y^2)^{-2} \begin{vmatrix} 1(x^2 + y^2) - x(2x), -2xy & 1(x^2 + y^2) - y(2y) \\ -2yx & x^2 - y^2 \end{vmatrix}$$

This has trace the sum of the diagonal elements, which is zero since $y^2 - x^2 + x^2 - y^2 = 0$.

The determinant is:

$$(x^2 + y^2)^{-4}((y^2 - x^2)(x^2 - y^2) - (-2xy)^2)$$

$$= (x^2 + y^2)^{-4}(-x^4 - y^4 - 2x^2y^2) = -(x^2 + y^2)^{-2}.$$

So the determinant is always negative and never vanishes.

- Show that the square of the derivative matrix $F'(X)$ is proportional to the identity matrix.

We compute the square of $F'(X)$ giving $(x^2 + y^2)^{-2}$ times the identity matrix, as required.
• Compute \((F \circ F)(X)\).

We have:
\[
(F \circ F)(X) = F(F(X)) = \frac{F(X)}{F(X) \cdot F(X)} = (X, X)^{-1}X \cdot (X, X)^{-2}X = X.
\]

• Use a suitable linear approximation to estimate \(F(3.1, 3.8)\).

We have with \(X = [3, 4]\) and \(A = [0.1, -0.2]\) the estimate:
\[
\begin{align*}
F([3, 4]) + F'(X)(A) &= \frac{[3, 4]}{3^2 + 4^2} + \frac{1}{10} (3^2 + 4^2)^{-2} \begin{bmatrix} 4^2 - 3^2 & -2(3)(4) \\ -2(4)(3) & 3^2 - 4^2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= \frac{[3, 4]}{25} + \frac{1}{6250} \begin{bmatrix} 7 & -24 \\ -24 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= \frac{1}{6250}([750, 1000] + [55, -10]) \\
&= \frac{1}{1250}([150, 200] + [11, -2]) \\
&= \frac{1}{1250}([161, 198]) = [0.1288, 0.1584].
\end{align*}
\]

Maple gives:
\[
F(3.1, 3.8) = [0.128898, 0.158004].
\]

So our estimate is pretty accurate.
Question 3

For \( X = [x, y, z] \neq [0, 0, 0] \), let a vector function \( G(X) \) be given by:

\[
G(X) = \frac{X}{|X|^3} = \frac{X}{(\mathbf{X} \cdot \mathbf{X})^{\frac{3}{2}}} = \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}[x, y, z].
\]

- Show that the trace of the derivative matrix \( G'(X) \) vanishes.

Put \( H(X) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \) and \( A = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \).

Then we have:

\[
H'(X)(A) = \begin{pmatrix} \nabla x & a \\ \nabla y & b \\ \nabla z & c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = A.
\]

Then we have:

\[
G'(X)(A) = (\mathbf{X} \cdot \mathbf{X})^{-\frac{3}{2}} H'(X)(A) + H(X)(\mathbf{X} \cdot \mathbf{X})^{-\frac{3}{2}} (2 \mathbf{X} \cdot A)
\]

\[
= (\mathbf{X} \cdot \mathbf{X})^{-\frac{3}{2}} A + X \left( -\frac{3}{2} \right) \mathbf{X} \cdot \mathbf{X}^{-\frac{3}{2}} (2 \mathbf{X} \cdot A)
\]

\[
= (\mathbf{X} \cdot \mathbf{X})^{-\frac{3}{2}} (\mathbf{X} \cdot \mathbf{A} - 3 (\mathbf{X} \cdot \mathbf{A}) \mathbf{X})
\]

\[
= (x^2 + y^2 + z^2)^{-\frac{3}{2}} \begin{pmatrix} x^2 - 3x^2 & -3xy & -3xz \\ -3xy & x^2 - 3y^2 & -3yz \\ -3xz & -3yz & x^2 - 3z^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}
\]

Note that we can get this same result (rather more quickly) by straightforward partial differentiation.

The trace is the sum of the diagonal elements, which is zero because

\[
(-2x^2 + y^2 + z^2) + (x^2 - 2y^2 + z^2) + (x^2 + y^2 - 2z^2) = 0.
\]

5
• Use a suitable linear approximation to estimate $G(2.1, 1.9, -0.7)$.

We put $X = [2, 2, -1]$ and $A = [2.1, 1.9, -0.7] - [2, 2, -1] = [0.1, -0.1, 0.3]$. Then we have $X \cdot X = 2^2 + 2^2 + (-1)^2 = 9$ and $X \cdot A = \frac{1}{10}[2, 2, -1].[1, -1, 3] = \frac{1}{10}[2 - 2 - 3] = -0.3$, so we get:

$$\Delta G = G'(X)(A) = (X \cdot X)^{-\frac{1}{2}}((X \cdot X)A - 3(X \cdot A)X)$$

$$= 9^{-\frac{1}{2}}(9A - 3(0.3)X)$$

$$= \frac{1}{27}(A + \frac{1}{10}X)$$

$$= \frac{1}{270}([3, 1, 2]).$$

Finally, we have $G(2, 2, -1) = \frac{[2, 2, -1]}{3^3} = \frac{1}{270}[20, 20, -10]$, so the required estimate is:

$$G(2, 2, -1) + \Delta G = \frac{1}{270}[23, 21, -8] = [0.0831, 0.0777, -0.0296].$$

Maple gives for $G(2.1, 1.9, -0.7)$ the value $[0.0846, 0.0765, -0.0281]$, so our estimate is not terribly accurate.
Question 4

A cylinder has radius $x$ and height $y$ meters.

- Compute the derivative of the map that assigns to the cylinder its volume and surface area.

We have, with $\mathbf{X} = [x, y]$, $\mathbf{A} = [a, b]$, volume $V$ and surface area $S$:

$$F(\mathbf{X}) = \frac{V}{S} = \pi \begin{vmatrix} x^2 y \\ 2x^2 + 2xy \end{vmatrix}$$

$$F'(\mathbf{X})(\mathbf{A}) = \begin{vmatrix} \frac{\nabla V}{\nabla S} \\ a \\ b \end{vmatrix} = \pi \begin{vmatrix} 2xy \\ 4x + 2y \\ a \\ b \end{vmatrix}$$

$$= \pi [2xya + x^2b, 4xa + 2ya + 2xb].$$

- Use your formula to estimate the changes in the volume and area, when $(x, y)$ changes from $(3, 4)$ to $(2.9, 4.3)$.

Putting in $\mathbf{X} = [3, 4]$ and $\mathbf{A} = \Delta \mathbf{X} = [2.9, 4.3] - [3, 4] = [-0.1, 0.3]$, we get the estimate:

$$\Delta F = \begin{vmatrix} \Delta V \\ \Delta S \end{vmatrix} = F'(\mathbf{X})(\mathbf{A})|_{\mathbf{X}=[3,4], \Delta=\frac{1}{10}[-0.1,0.3]}

= \frac{1}{10} \begin{vmatrix} 2(3(4)) & 3^2 \\ 4(3) + 2(4) & 2(3) \end{vmatrix} \begin{vmatrix} -1 \\ 3 \end{vmatrix}

= \frac{1}{10} \begin{vmatrix} 24 & 9 \\ 20 & 6 \end{vmatrix} \begin{vmatrix} -1 \\ 3 \end{vmatrix}

= \frac{1}{10} \begin{vmatrix} -24 + 27 \\ -20 + 18 \end{vmatrix}

= \begin{vmatrix} -0.3 \\ -0.2 \end{vmatrix}.$$
Question 5

A rectangular block has length $x$, breadth $y$ and height $z$ meters.

- Compute the derivative of the map that assigns to the block its volume, surface area and total edge length.

- Use your formula to estimate the changes in these quantities, when $(x, y, z)$ changes from $(3, 4, 5)$ to $(2.8, 3.7, 4.4)$.

We have, with $\mathbf{X} = [x, y, z]$, $\mathbf{A} = [a, b, c]$, volume $V$, surface area $S$ and total edge length $P$:

$$F(\mathbf{X}) = \begin{vmatrix} V \\ S \\ P \end{vmatrix} = \begin{vmatrix} xyz \\ 2(yz + zx + xy) \\ 4(x + y + z) \end{vmatrix}$$

$$F'(\mathbf{X})(\mathbf{A}) = \begin{vmatrix} \nabla V \\ \nabla S \\ \nabla P \end{vmatrix} \begin{vmatrix} a \\ b \\ c \end{vmatrix} = \begin{vmatrix} yz \\ zx \\ xy \\ 2(y + z) \\ 2(z + x) \\ 2(x + y) \end{vmatrix} \begin{vmatrix} a \\ b \\ c \end{vmatrix} = \begin{vmatrix} ayz + bzx + cxy \\ 2ay + 2az + 2bx + 2cy + 2cx \end{vmatrix}.$$   

Putting in $\mathbf{X} = [3, 4, 5]$ and $\mathbf{A} = \Delta \mathbf{X} = [2.8, 3.7, 4.4] - [3, 4, 5] = [-0.2, -0.3, -0.6]$, we get the estimate:

$$\Delta F = \begin{vmatrix} \Delta V \\ \Delta S \\ \Delta P \end{vmatrix} = F'(\mathbf{X})(\mathbf{A})|_{\mathbf{X}=[3,4,5],\Delta=\frac{-1}{10}[2.3,6]} = -\frac{1}{10} \begin{vmatrix} 4(5) \\ 5(3) \\ 3(4) \end{vmatrix} = \begin{vmatrix} 2(4 + 5) \\ 2(5 + 3) \\ 2(3 + 4) \end{vmatrix} = \begin{vmatrix} 2 \\ 3 \\ 6 \end{vmatrix}$$

$$= -\frac{1}{10} \begin{vmatrix} 20 \\ 18 \end{vmatrix} = -\frac{1}{10} \begin{vmatrix} 15 \\ 16 \end{vmatrix} = -\frac{1}{10} \begin{vmatrix} 12 \\ 14 \end{vmatrix} = \begin{vmatrix} 2 \\ 3 \\ 6 \end{vmatrix} = \begin{vmatrix} 40 + 45 + 24 \\ 36 + 48 + 84 \\ 8 + 12 + 24 \end{vmatrix} = \begin{vmatrix} 109 \\ 168 \end{vmatrix} = \begin{vmatrix} 109 \\ -10.9 \end{vmatrix} = \begin{vmatrix} -4.4 \end{vmatrix}$$