Calculus III   Homework 4, Solutions, due 9/28/5

Question 1

Let \( A = [3, 2, -2], B = [6, -1, -11], C = [1, 5, 0] \) and \( D = [-2, 11, -3] \) be points in space.

- Show that the lines \( AB \) and \( CD \) meet and find their point of intersection.

We have:

\[
AB = B - A = [6, -1, -11] - [3, 2, -2] = [3, -3, -9],
\]

\[
CD = D - C = [-2, 11, -3] - [1, 5, 0] = [-3, 6, -3],
\]

The line \( AB \) is then:

\[
X = A + sAB = [3, 2, -2] + s[3, -3, -9] = [3 + 3s, 2 - 3s, -2 - 9s],
\]

\[
x = 3 + 3s, \quad y = 2 - 3s, \quad z = -2 - 9s.
\]

The line \( CD \) is:

\[
X = C + sCD = [1, 5, 0] + t[-3, 6, -3] = [1 + 3t, 5 + 6t, -3t].
\]

The two lines meet where both sets of equations hold simultaneously:

\[
3 + 3s = 1 - 3t,
\]

\[
2 - 3s = 5 + 6t,
\]

\[
-2 - 9s = -3t.
\]

Adding the first two equations gives:

\[
5 = 6 + 3t, \quad 3t = -1, \quad t = -\frac{1}{3}.
\]

Back substituting into the first equation gives:

\[
3 + 3s = 1 - 3(-\frac{1}{3}) = 1 + 1 = 2, \quad 3s = -1, \quad s = -\frac{1}{3}.
\]

For \( s = -\frac{1}{3} \) the point on \( AB \) is:

\[
X = [3+3(-\frac{1}{3}), 2-3(-\frac{1}{3}), -2-9(-\frac{1}{3})] = [3-1, 2+1, -2+3] = [2, 3, 1].
\]

For \( t = -\frac{1}{3} \) the point on \( CD \) is:

\[
X = [1 - 3(-\frac{1}{3}), 5 + 6(-\frac{1}{3}), -3(-\frac{1}{3})] = [1 + 1, 5 - 2, 1] = [2, 3, 1].
\]

So the lines meet at \([2, 3, 1]\).
• Find the equation of the plane containing the lines \( AB \) and \( CD \).

A vector perpendicular to the plane is:

\[
\mathbf{AB} \times \mathbf{CD} = [3, -3, -9] \times [-3, 6, -3]
\]

\[
= \det \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
3 & -3 & -9 \\
-3 & 6 & -3
\end{vmatrix}
= \hat{i} \begin{vmatrix}
-3 & -9 \\
6 & -3
\end{vmatrix} - \hat{j} \begin{vmatrix}
3 & -9 \\
-3 & -3
\end{vmatrix} + \hat{k} \begin{vmatrix}
3 & -3 \\
-3 & 6
\end{vmatrix}
= \hat{i}(9 + 54) - \hat{j}(-9 - 27) + \hat{k}(18 - 9)
= 63\hat{i} + 36\hat{j} + 9\hat{k} = 9(7\hat{i} + 4\hat{j} + \hat{k}).
\]

After scaling down (dividing) by a common factor of 9, we may take the normal \( \mathbf{N} \) to be \([7, 4, 1]\).

Then the plane has equation:

\[
7(x - 2) + 4(y - 3) + (z - 1) = 0,
\]

\[
7x + 4y + z - 14 - 12 - 1 = 0,
\]

\[
7x + 4y + z = 27.
\]

It is easily checked that all the five points \( A, B, C, D \) and \( P \) lie on this plane, as required.

• Find the angle between the lines \( AB \) and \( CD \).

The required angle is the angle between the direction vectors of the lines which are the vectors \( \mathbf{AB} = [3, -3, -9] \) and \( \mathbf{CD} = [-3, 6, -3] \).

We have:

\[
\mathbf{AB} \cdot \mathbf{CD} = [3, -3, -9] \cdot [-3, 6, -3] = -9 - 18 + 27 = 0.
\]

So the lines \( AB \) and \( CD \) are perpendicular.
• Find the area of the quadrilateral $ABDC$.

Let $P = [2, 3, 1]$ be the point of intersection of the lines $AB$ and $CD$, found earlier.

Then the ordering of the points on the line $AB$ is $PAB$ (compare their $x$-co-ordinates: 2, 3, 6) and the ordering of the points of the line $CD$ is $PCD$ (again compare their $x$-co-ordinates: 2, 1, -2).

So the area of the quadrilateral is $\alpha - \beta$ where $\alpha$ is the area of the outer triangle $PBD$ and $\beta$ is the area of the inner triangle $PAC$.

We have, since each of these triangles are right-angled at $P$:

$$PA = A - P = [3, 2, -2] - [2, 3, 1] = [1, -1, -3],$$

$$|PA| = ||[1, -1, -3]| = \sqrt{1^2 + (-1)^2 + 3^2} = \sqrt{1 + 1 + 9} = \sqrt{11},$$

$$PB = B - P = [6, -1, -11] - [2, 3, 1] = [4, -4, -12] = 4PA,$$

$$|PB| = 4|PA| = 4\sqrt{11},$$

$$PC = C - P = [1, 5, 0] - [2, 3, 1] = [-1, 2, -1],$$

$$|PC| = ||[-1, 2, -1]| = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{1 + 4 + 1} = \sqrt{6},$$

$$PD = D - P = [-2, 11, -3] - [2, 3, 1] = [-4, 8, -4] = 4PC,$$

$$|PD| = 4|PC| = 4\sqrt{11},$$

$$\beta = \frac{1}{2}|PA||PC| = \frac{1}{2}\sqrt{11}\sqrt{6} = \frac{1}{2}\sqrt{66},$$

$$\alpha = \frac{1}{2}|PB||PD| = 4^2 \left( \frac{1}{2}|PA||PC| \right) = 16\beta,$$

$$\alpha - \beta = 15\beta = \frac{15}{2}\sqrt{66}. $$

So the area of the quadrilateral $ABDC$ is $\frac{15}{2}\sqrt{66} = 60.93029$ square units.
Question 2

A hyperbola of eccentricity 2 has a focus at the point (3, 3). Its directrix is the line \( y = -x \). Find the equation of the hyperbola and sketch the hyperbola.

If \( F \) is the focus, \( L \) is the directrix, \( \epsilon \) is the eccentricity and \( P = (x, y) \) is a point of the parabola, then we need \( PF = \epsilon PL \).

Here we have:

- \( \epsilon = 2 \).
- \( F = (3, 3), PF = \sqrt{(x - 3)^2 + (y - 3)^2} \).
- In two dimensions, the distance of a point \( X = (x, y) \) from a line with equation \( N \cdot X = c \) is \( \frac{|N \cdot X - c|}{|N|} \).

Here the line \( L \) has the equation \( x + y = 0 \), or \([1, 1].[x, y] = 0\), or \( N \cdot X = 0\), where \( N = [1, 1] \) is the normal to the line. Then we have:

- \( c = 0 \),
- \( |N| = ||[1, 1]| = \sqrt{1^2 + 1^2} = \sqrt{1 + 1} = \sqrt{2} \).

So the distance of \( P \) from \( L \) is:

\[
PL = \frac{|N \cdot X - c|}{|N|} = \left| \frac{x + y - 0}{\sqrt{2}} \right| = \left| \frac{x + y}{\sqrt{2}} \right|.
\]

So the hyperbola has the equation:

\[
PF = 2PL, \quad \sqrt{(x - 3)^2 + (y - 3)^2} = 2 \left| \frac{x + y}{\sqrt{2}} \right|
\]

\[
(x - 3)^2 + (y - 3)^2 = 4 \left( \frac{x + y}{2} \right)^2 = 2(x + y)^2,
\]

\[
x^2 - 6x + 9 + y^2 - 6y + 9 = 2x^2 + 4xy + 2y^2,
\]

\[
x^2 + 4xy + y^2 + 6x + 6y - 18 = 0.
\]
We can factor the quadratic terms:
\[ x^2 + 4xy + y^2 = x^2 + 4xy + 4y^2 - 3y^2 = (x + 2y)^2 - 3y^2 = (x + 2y)^2 - (y\sqrt{3})^2 \]
\[ = (x + 2y + y\sqrt{3})(x + 2y - y\sqrt{3}) = (x + (2 + \sqrt{3})y)(x + (2 - \sqrt{3})y). \]

Now expand out the following expression:
\[ (x + (2 + \sqrt{3})y + a)(x + (2 - \sqrt{3})y + b) - C \]
\[ = x^2 + 4xy + y^2 + (a+b)x + (a(2 - \sqrt{3}) + b(2 + \sqrt{3}))y + ab - C. \]

This agrees with the equation of the hyperbola if:
\[ a + b = 6, \quad a(2 - \sqrt{3}) + b(2 + \sqrt{3}) = 6, \quad ab - C = -18, \]
\[ a(2 - \sqrt{3} - 2 - \sqrt{3}) + 12 + 6\sqrt{3} = 6, \]
\[ -2\sqrt{3}a = -6 - 6\sqrt{3}, \quad -6a = -6\sqrt{3} - 6(3), \]
\[ a = \sqrt{3} + 3, \quad b = 6 - a = 3 - \sqrt{3}, \]
\[ C = ab + 18 = (3 + \sqrt{3})(3 - \sqrt{3}) + 18 = 9 - 3 + 18 = 24. \]

So we may rewrite the equation of the hyperbola as:
\[ (x + (2 + \sqrt{3})y + 3 + \sqrt{3})(x + (2 - \sqrt{3})y + 3 - \sqrt{3}) = 24. \]

Then the asymptotes are the lines:
\[ 0 = x + (2 + \sqrt{3})y + 3 + \sqrt{3}, \quad 0 = x + (2 - \sqrt{3})y + 3 - \sqrt{3}. \]

These meet where \(2\sqrt{3}y + 2\sqrt{3} = 0\), so where \(y = -1\) and then \(x = -(2 + \sqrt{3})(-1) - 3 - \sqrt{3} = -1.\)

So the center of the hyperbola is at \((-1, -1)\) and the asymptotes are the two lines through the center of slopes:
\[ -\frac{1}{2 + \sqrt{3}} = -\frac{2 - \sqrt{3}}{2^2 - (\sqrt{3})^2} = \sqrt{3} - 2, \]
\[ -\frac{1}{2 - \sqrt{3}} = -\frac{2 + \sqrt{3}}{2^2 - (\sqrt{3})^2} = -\sqrt{3} - 2. \]

The hyperbola is symmetric under the interchange of \(y\) and \(x\), so under reflection about the line \(y = x\), so by symmetry the other focus is on the line \(y = x\) and is equidistant from the center \((-1, -1)\) as is the focus \((3, 3)\), so is at the point \((-5, -5)\).

Finally the hyperbola crosses its axis where \(y = x\), so where \(6x^2 + 12x - 18 = 0\), or \(x^2 + 2x - 3 = 0\), or \((x + 3)(x - 1) = 0\), so at the points \((-3, -3)\) and \((1, 1)\).
**Question 3**

Find the symmetric and parametric equations of the line of intersection of the planes: \(2x - y + 2z = -2\) and \(3x + 4y - 5z = 13\). Also find the angle between the planes.

We take \(z = t\) as the parameter and solve for \(x\) and \(y\).

We have:

\[
\begin{align*}
2x - y &= -2 - 2z = -2 - 2t, \\
3x + 4y &= 13 + 5z = 13 + 5t.
\end{align*}
\]

Multiplying the first equation by 4 and adding to the second equation gives:

\[
\begin{align*}
8x - 4y + 3x + 4y &= -8 - 8t + 13 + 5t = 5 - 3t \\
11x &= 5 - 3t, \quad x = \frac{1}{11}(5 - 3t), \\
y &= 2x + 2z + 2 = \frac{1}{11}(10 - 6t + 22t + 22) = \frac{1}{11}(32 + 16t).
\end{align*}
\]

So the required parametric equations are:

\[
[x, y, z] = \frac{1}{11}[5 - 3t, 32 + 16t, 11t].
\]

Solving for \(t\), we get the symmetric equations as:

\[
\begin{align*}
\frac{11x - 5}{-3} &= \frac{11y - 32}{16} = z.
\end{align*}
\]

One plane has normal \(\overrightarrow{M} = [2, -1, 2]\), the other has normal \(\overrightarrow{N} = [3, 4, -5]\). We have, if \(\theta\) is the required angle:

\[
\begin{align*}
\overrightarrow{M} \cdot \overrightarrow{N} &= [2, -1, 2] \cdot [3, 4, -5] = 6 - 4 - 10 = -8, \\
|\overrightarrow{M}| &= \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3, \\
|\overrightarrow{N}| &= \sqrt{3^2 + 4^2 + (-5)^2} = \sqrt{9 + 16 + 25} = \sqrt{50} = 5\sqrt{2}, \\
\cos(\theta) &= \frac{\overrightarrow{M} \cdot \overrightarrow{N}}{|\overrightarrow{M}| |\overrightarrow{N}|} = \frac{-8}{3(5\sqrt{2})} = -\frac{4\sqrt{2}}{15}, \\
\theta &= \arccos\left(-\frac{4\sqrt{2}}{15}\right) = 1.957484953 \text{ radians} = 112.1556262 \text{ degrees}.
\end{align*}
\]
**Question 4**

Let \( f(x, y) = x^2 + xy - 2y^2 - 7x + y + 12 \). 

- Find the equation of the tangent plane to the surface \( z = f(x, y) \) at the point \([2, 2, 0]\)
  
  We have:
  
  \[
  \frac{\partial z}{\partial x}|_{[2,2,0]} = (2x + y - 7)|_{[2,2,0]} = 2(2) + 2 - 7 = 4 + 2 - 7 = -1,
  \]
  
  \[
  \frac{\partial z}{\partial y}|_{[2,2,0]} = (x - 4y + 1)|_{[2,2,0]} = 2 - 4(2) + 1 = 2 - 8 + 1 = -5,
  \]

  So vectors lying in the plane are:
  
  \[ V = [1, 0, \frac{\partial z}{\partial x}]_{[2,2,0]} = [1, 0, -1], \]
  \[ W = [0, 1, \frac{\partial z}{\partial y}]_{[2,2,0]} = [0, 1, -5], \]

  Then the plane has normal \( N \):
  
  \[
  N = V \times W = \text{det} \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -5 \end{vmatrix} = i \begin{vmatrix} 0 & -1 \\ 1 & -5 \end{vmatrix} - j \begin{vmatrix} 1 & -1 \\ 0 & -5 \end{vmatrix} + k \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = i(0 + 1) - j(-5 - 0) + k(1 - 0) = i + 5j + k.
  \]

  Since the plane goes through the point \( A = [2, 2, 0] \) it has the equation:
  
  \[
  0 = N.(X - A) = [1, 5, 1].[x - 2, y - 2, z - 0] = x - 2 + 5(y - 2) + z,
  \]
  
  \[
  x + 5y + z - 12 = 0.
  \]
Find all points of the surface \( z = f(x, y) \), for which the tangent plane is horizontal.

The tangent plane is horizontal if both the partial derivatives \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) vanish, so if:

\[
2x + y - 7 = 0, \quad x - 4y + 1 = 0.
\]

Multiplying the first equation by 4 and adding to the second equation gives:

\[
0 = 8x + 4y - 28 + x - 4y + 1 = 9x - 27,
\]

\[
x = 3, \quad y = 7 - 2x = 7 - 2(3) = 7 - 6 = 1.
\]

Putting \( x = 3 \) and \( y = 1 \), so we see that both partial derivatives do indeed vanish.

Also we get \( z = 3^2 + 3(1) - 2(1)^2 - 7(3) + 1 + 12 = 9 + 3 - 2 - 21 + 1 + 12 = 2 \).

So the only point where the tangent plane is horizontal is the point \((3, 1, 2)\).

Discuss the shapes of the level curves with \( z \) constant on the surface with equation:

\( z = f(x, y) \).

If \( z = A \), we have a quadratic equation:

\[
x^2 + xy - 2y^2 - 7x + y + (12 - A) = 0.
\]

This is a conic; in general the conic \( ax^2 + bxy + cy^2 + px + qy + r = 0 \) (with \( a, b \) and \( c \) not all zero) is:

- A parabola or a repeated line, iff \( b^2 = 4ac \) iff the quadratic terms \( ax^2 + bxy + cy^2 \) are a constant multiple of a perfect square.
  
  So for example \( 2x^2 - 8xy + 8y^2 + 6x + 10 - 11 \) is a parabola.
  
  The degenerate case of a repeated line occurs only if the whole equation is a perfect square.

- An ellipse, circle or point or empty set, iff \( b^2 - 4ac < 0 \) iff the quadratic equation \( ax^2 + bxy + cy^2 \) has no real roots for the ratio \( y : x \).
  
  So, for example the curve \( x^2 + xy + y^2 - 7x + y = 0 \) has \( b = 1, a = 1, \)
  \( c = 1, \) so \( b^2 - 4ac = 1^2 - 4(1(1) = 1 - 4 = -3 < 0, \) so represents an ellipse or circle, since we can see that it has at least two points: \( x = 0 \) and \( y = 0 \) or \( y = 7 \).
  
  It is an ellipse, since the circle case can occur only if \( b = 0 \) and \( a = c \neq 0 \).
- A hyperbola or a line pair, iff \( b^2 - 4ac > 0 \) iff the quadratic equation 
\[
ax^2 + bxy + cy^2 = 0
\]
has two real roots for the ratio \( y : x \); these ratios give the slopes of the asymptotes; so, for example the curve 
\[
x^2 + xy - 2y^2 - 7x + y = 0
\]
has \( b = 1, a = 1, c = -2 \), so \( b^2 - 4ac = 1^2 - 4(1)(-2) = 9 > 0 \), so represents a hyperbola.

The degenerate case of a line pair occurs only if the whole equation factorizes.

The factorization of the quadratic terms is 
\[
x^2 + xy - 2y^2 = (x + 2y)(x - y).
\]
So the asymptotes have slopes 1 (from \( x - y \)) and \(-\frac{1}{2}\) (from \( x + 2y \)).

- In the present case, the level surfaces are hyperbolas:
\[
x^2 + xy - 2y^2 - 7x + y + C = 0, \quad C = 12 - A.
\]

We factor the quadratic terms and rewrite the equation as follows: Consider the following expansion:
\[
0 = x^2 + xy - 2y^2 - 7x + y + C = (x - y + a)(x + 2y + b) + c
\]
Expanding out the right-hand side, we get:
\[
x^2 + xy - 2y^2 - 7x + y + C = x^2 + xy - 2y^2 + (a + b)x + (2a - b)y + ab + c.
\]
Comparing coefficients, we get \( a + b = -7, 2a - b = 1 \) and \( C = ab + c \).

Adding the first two equations eliminates \( b \) and gives \( 3a = -6 \), so \( a = -2 \), then \( b = 2a - 1 = 2(-2) - 1 = -5 \).

Then \( c = C - ab = 12 - A - (-2)(-5) = 2 - A \).

So the equation is now \((x - y - 2)(x + 2y - 5) = A - 2\).

This is a line pair if \( A = 2 \) and a hyperbola for all other values of \( A \); the asymptotes are the line pair: \( x - y - 2 = 0 \) and \( x + 2y - 5 = 0 \); these lines are of slopes 1 and \(-\frac{1}{2}\), respectively, as expected and meet at the center of the hyperbola where both equations hold at once.

Subtracting the equations gives: \(-3y + 3 = 0\), so \( y = 1 \) and then \( x = y + 2 = 1 + 2 = 3 \).

So the hyperbolas are centered at \((3, 1)\).

If we put \( x = 4 \) and \( y = 1 \), we have \((x - y - 2)(x + 2y - 5) = (4 - 1 - 1)(4 + 2 - 5) > 0\).
So if \( A > 2 \), one branch of the hyperbola lies between the asymptotes in the same region as the point \((4, 1)\) and if \( A < 2 \) no branch does.
**Question 5**

Find a function $g(x, y)$ such that $\frac{\partial g}{\partial x} = -\frac{y}{x^2 + y^2}$ and $\frac{\partial g}{\partial y} = \frac{x}{y} \frac{1}{x^2 + y^2}$.

How do the domains of your function $g$ and of the functions $-\frac{y}{x^2 + y^2}$ and $\frac{x}{y} \frac{1}{x^2 + y^2}$ compare? Explain your answer.

We want $\frac{\partial g}{\partial x} = -\frac{y}{x^2 + y^2}$.

Since the derivative keeps $y$ fixed, we should be able to get $g$ by integrating the right-hand side with respect to $x$, regarding $y$ as a constant:

$$g = \int -\frac{y}{x^2 + y^2} \, dx = \int -\frac{d}{dx} x^2 + y^2 \, dx = -y \int \frac{dx}{x^2 + y^2}.$$

Recall the integration formula: $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \left( \frac{x}{a} \right) + C$.

So here $a = y$ and we get:

$$g = \int \frac{-y}{x^2 + y^2} \, dx = -y \int \frac{dx}{x^2 + y^2} = -y \arctan \left( \frac{x}{y} \right) + C = -\arctan \left( \frac{x}{y} \right) + C.$$

Note that the "constant of integration" here is at fixed $y$, so $C$ can depend on $y$, so our candidate for $g$ is:

$$g = -\arctan \left( \frac{x}{y} \right) + C(y).$$

Differentiating with respect to $x$, we get:

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x} \left( -\arctan \left( \frac{x}{y} \right) \right) = -\frac{1}{1 + \left( \frac{x}{y} \right)^2} \frac{\partial}{\partial x} \left( \frac{x}{y} \right)$$

$$= -\frac{y^2}{y^2 + x^2} \left( \frac{1}{y} \right) = -\frac{y}{y^2 + x^2}.$$
Differentiating with respect to $y$, we get:

$$
\frac{\partial g}{\partial y} = \frac{\partial}{\partial y} \left( -\arctan \left( \frac{x}{y} \right) \right) + C'(y) = -\frac{1}{1 + \left( \frac{x}{y} \right)^2} \frac{\partial}{\partial y} \left( \frac{x}{y} \right) + C'(y)
$$

$$
= -\frac{y^2}{y^2 + x^2} \left( -\frac{x}{y^2} \right) + C'(y) = \frac{x}{y^2 + x^2} + C'(y).
$$

So the function $g$ has the correct partial derivatives provided $C'(y) = 0$, so provided $C(y)$ is a constant.

So we may write our solution as $g = -\arctan \left( \frac{x}{y} \right) + C$, with $C$ an arbitrary constant.

Note that the domain of our solution $g$ is all $(x, y)$ in the plane with $y \neq 0$, so everywhere in the plane except along the $x$-axis.

Then $g$ is defined on a smaller domain than the functions $\frac{x}{x^2 + y^2}$ and $\frac{y}{x^2 + y^2}$, both of which are defined for all $(x, y)$ away from the origin.

We can find another solution $g = \arctan \left( \frac{y}{x} \right)$ which is defined away from the $y$-axis, but it can be shown that we can never find a solution which is defined on all points away from the origin.