The matrix for rotation through an angle $t$ radians counter-clockwise about the origin is:

$$R(t) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$

- Multiply $R(s)$ and $R(t)$ and use trigonometric identities to show that $R(s)R(t) = R(t)R(s) = R(s + t)$ and that the inverse of $R(t)$ is $R(-t)$.

The action of $R(t)$ on the point $X = (x, y)$ (written as a column) is given by:

$$X \rightarrow X' = R(t)X$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Let $R = \begin{bmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{-4}{5} & \frac{3}{5} \end{bmatrix}$.
  What is the angle of rotation of $R$?
  Find the image $A'B'C'$ under $R$ of the triangle $ABC$ with vertices:
  Also compute the lengths of the sides of the triangle $ABC$ and of $A'B'C'$ and show that the two triangles are congruent.

- Let $S = \begin{bmatrix} \frac{1}{4} & \sqrt{2} + \sqrt{6} \\ \frac{1}{4} & \sqrt{2} - \sqrt{6} \end{bmatrix}$.
  Find the smallest positive integer $n$ such that $S^n$ is the identity matrix.
  Also show that $S$ is a rotation matrix and identify the angle of rotation.
When we want to rotate about an axis that is not the origin, we represent points by \( X = (x, y, 1) \) (written as a column) and the rotation is then a matrix of the following form:

\[
R(t) = \begin{pmatrix}
\cos(t) & -\sin(t) & a \\
\sin(t) & \cos(t) & b \\
0 & 0 & 1
\end{pmatrix}.
\]

If \( a = b = 0 \) this corresponds to an ordinary rotation about the origin. A translation matrix is a matrix of the following form:

\[
T = \begin{pmatrix}
1 & 0 & a \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix}.
\]

Show that the general rotation matrix is the product of a rotation about the origin and a translation.

Let \( U = \begin{pmatrix} 3 & -4 & 2 \\ 4 & 5 & -2 \\ 0 & 0 & 1 \end{pmatrix} \).

What is the angle of rotation of \( U \)?

Find the image \( A''B''C'' \) under \( U \) of the triangle \( ABC \) with vertices: \( A = [4, 3], B = [2, -1] \) and \( C = [3, -4] \).

Also compute the lengths of the sides of the image triangle and show that the two triangles are congruent.

Also show that \( A''B''C'' \) is just a translate of \( A'B'C' \) and identify the translation involved.

Sketch all three triangles.

The center of a rotation \( R \) is a point \( X \) which is left invariant under the action of the rotation, so it obeys the equation \( RX = X \).

- For the rotation \( U \) given above, find its center of rotation.
- What is the center of rotation of \( U^2 \).

Explain.
A general conic is given by the equation:

\[ ax^2 + 2bxy + cy^2 + 2px + 2qy + r = 0. \]

This is nicely represented in matrix form as follows:

\[
X^T C X = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a & b & p \\ b & c & q \\ p & q & r \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
\]

Verify this by expanding out the right-hand side. The image under a rotation \( R \) of a conic \( C \) is given by the formula:

\[ C \rightarrow RCR^T. \]

- Find and sketch the image under the rotations \( R \) and \( U \) given above of each of the following:
  - The circle with equation \( x^2 + y^2 = 1 \).
  - The circle center \((3,4)\) of radius 5.
  - The parabola \( 3x^2 + 10xy + 3y^2 + 4x + 8y + 2 = 0 \).
  - The hyperbola \( x^2 + 4xy + y^2 - 6x + 8y = 0 \).
  - The ellipse \( 4x^2 + 2xy + 4y^2 - 10x - 10y = 0 \).

In each case verify that the image of the conic is another conic of the same type.