Honors Calculus Quiz 9 Solutions 12/2/5

Question 1
Find the centroid of the region $\mathcal{R}$ bounded by the curves $10y - y^2 + 2 = x$ and $2y^2 - 20y + 50 = x$.
Also determine the volumes of revolution of the region $\mathcal{R}$ about the coordinate axes.

We first find where the two parabolas meet:

\begin{align*}
10y - y^2 + 2 &= 2y^2 - 20y + 50, \\
0 &= 3y^2 - 30y + 48, \\
0 &= y^2 - 10y + 16, \\
0 &= (y - 2)(y - 8).
\end{align*}

So $y = 2, x = 18$ or $y = 8, x = 18$.

Plotting we see the region $\mathcal{R}$ lies between the two parabolas with the curve $x_- = 2y^2 - 20y + 50$ on the left and $x_+ = 10y - y^2 + 2$ on the right.
Also we see that the region is symmetrical about the line $y = 5$, so the centroid $C$ of $\mathcal{R}$ lies on that line, so $C = [X, 5]$ for some $X$.
To verify the symmetry we put $y = t + 5$, then we have:

\begin{align*}
x_+ &= 10y - y^2 + 2 = 10(t + 5) - (t + 5)^2 + 2 \\
&= 10t + 50 - (t^2 + 10t + 25) + 2 = 27 - t^2, \\
x_- &= 2y^2 - 20y + 50 = 2(t + 5)^2 - 20(t + 5) + 50 \\
&= 2(t^2 + 10t + 25) - 20t - 100 + 50 \\
&= 2t^2.
\end{align*}

Since clearly both of these parabolas are symmetrical under $t \to -t$, the original graphs are symmetrical about the line $y = 5$, as required.
The area $A$ of $R$ is, using horizontal strips:

$$A = \int_2^8 (x_+ - x_-)dy$$

$$= \int_2^8 ((10y - y^2 + 2) - (2y^2 - 20y + 50))dy$$

$$= \int_2^8 (30y - 3y^2 - 48)dy$$

$$= [15y^2 - y^3 - 48y]_2^8 = 15(8^2) - 8^3 - 48(8) - (15(2^2) - 2^3 - 48(2))$$

$$= 8^2(15 - 8 - 6) - (60 - 8 - 96)$$

$$= 64 + 44 = 108.$$

If $V_x$ is the volume obtained by rotating about the $x$-axis and $V_y$ is the volume obtained by rotating around the $y$-axis, then we have by Pappus’ Theorem:

$$[V_y, V_x] = 2\pi AC = 216\pi [X, 5],$$

$$V_y = 216\pi X,$$

$$V_x = 1080\pi = 3392.92.$$

Finally, using horizontal washers, we get for $V_y$:

$$V_y = \pi \int_2^8 (x_+^2 - x_-^2)dy = \pi \int_{-3}^3 ((27 - t^3) - (2t^2)^2)dt$$

$$= \pi \int_{-3}^3 (729 - 54t^2 - 3t^4)dt = 6\pi \int_{0}^3 (243 - 18t^2 - t^4)dt$$

$$= 6\pi \left[ 243t - 6t^3 - \frac{t^5}{5} \right]_0^3 = 6\pi(3) \left( 243 - 6(9) - \frac{3^4}{5} \right)$$

$$= 6\pi(27) \left( 27 - 6 - \frac{9}{5} \right) = \frac{1}{5} 6\pi(27)(105 - 9) = \frac{\pi}{5}(15552) = 9771.61$$

Then we have:

$$X = \frac{V_y}{216\pi} = \frac{15552\pi}{5(216\pi)} = \frac{2592}{5(36)} = \frac{432}{5(6)} = \frac{72}{5} = 14.4.$$ 

So the centroid of the region $R$ is at $C = \frac{1}{5}[72, 25] = [14.4, 5]$. 

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Question 2

Discuss the convergence of the following series and see if you can find a formula for the sum explicitly when the series converges.

- \[ \sum_{n=1}^{\infty} \frac{1}{n(n+2)} \]

We use partial fractions:

\[ \frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}, \]

\[ 1 = A(n + 2) + B(n), \]

\[ 1 = -2B, \]

\[ 1 = 2A, \]

\[ \frac{1}{n(n+2)} = \frac{1}{2} \left( \frac{1}{n} - \frac{1}{n+2} \right). \]

Writing out the series we get for the first \( n \) terms:

\[ \frac{1}{2} \left( \left( 1 - \frac{1}{3} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \left( \frac{1}{4} - \frac{1}{6} \right) + \cdots + \left( \frac{1}{n} - \frac{1}{n+2} \right) \right). \]

This telescopes, with each negative term cancelling with the next but one positive term, so the \( n \)-partial sum, \( s_n \) is:

\[ s_n = \frac{1}{2} \left( 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) \]

\[ = \frac{1}{4(n+1)(n+2)}(3(n+1)(n+2) - 2(n+2) - 2(n+1)) \]

\[ = \frac{1}{4(n+1)(n+2)}(3n^2 + 9n + 6 - 2n - 4 - 2n - 2) \]

\[ = \frac{n(3n+5)}{4(n+1)(n+2)}. \]

Then the series sum \( s \) is the limit of \( s_n \) as \( n \to \infty \) so is \( s = \frac{3}{4} \).
\[
\sum_{n=1}^{\infty} \frac{n4^n}{3^{2n}}
\]

The ratio test gives:
\[
a_n = \frac{n4^n}{3^{2n}}, \quad a_{n+1} = \frac{(n+1)4^{n+1}}{3^{2(n+1)}}
\]
\[
r_n = \frac{a_{n+1}}{a_n} = \frac{(n+1)4^{n+1}}{3^{2(n+1)}} \cdot \frac{3^{2n}}{n4^n} = \frac{4(n+1)}{9n}.
\]

As \(n \to \infty\), we get the limiting ratio \(|r_n| \to r = \frac{4}{9}\), which is less than 1, so the series converges by the ratio test.

For the sum, see below.

\[
\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{4^n(2n + 1)!}
\]

The standard series of \(\sin(y)\) is, valid for any \(y\) (real or complex, in fact):
\[
\sin(y) = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n + 1)!}.
\]

Substitute \(y = \frac{x}{2}\) and we get:
\[
\sin \left(\frac{x}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}(2n + 1)!}.
\]

Since \(2^{2n+1} = 2(4^n)\), this is one-half of the given sum, so the given sum is \(2 \sin \left(\frac{x}{2}\right)\), for any \(x\).
\begin{itemize}
    \item \( f(x) = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{2^n} \).
\end{itemize}

Define \( G(y) \) by the formula:
\[
G(y) = \sum_{n=0}^{\infty} (n+1)y^n = 1 + 2y + 3y^2 + 4y^3 + \ldots.
\]

Then the given series is \( f(x) = G \left( \frac{x}{2} \right) \).

Note that if \( |y| \geq 1 \), the individual terms in the series \( G(y) \) have sizes larger than \( n \), which goes to infinity as \( n \) goes to infinity, so the series diverges.

So we may assume that \( |y| < 1 \).

We see that for \( y \neq 1 \), the \( n \)-th partial sum is:
\[
G_n(y) = 1 + 2y + 3y^2 + \cdots + ny^{n-1}
= \frac{d}{dy} (1 + y + y^2 + \cdots + y^n)
= \frac{d}{dy} \left( \frac{y^{n+1} - 1}{y - 1} \right)
= \frac{1}{(y-1)^2} \left( (n+1)y^n(y-1) - 1(y^{n+1} - 1) \right)
= \frac{1}{(y-1)^2} (ny^{n+1} - (n+1)y^n + 1).
\]

Since \( |y| < 1 \), the terms \( ny^{n+1} \) and \( (n+1)y^n \) go to zero as \( n \) goes to infinity, so the series converges with sum:
\[
G(y) = \lim_{n \to \infty} \frac{1}{(y-1)^2} (ny^{n+1} - (n+1)y^n + 1) = \frac{1}{(1-y)^2}.
\]

So the series for \( f(x) \) converges iff \( |x| < 2 \), so on the open interval \((-2,2)\), with limit:
\[
f(x) = G \left( \frac{x}{2} \right) = \frac{1}{\left( 1 - \frac{x}{2} \right)^2} = \frac{4}{(2-x)^2}.
\]
We return to the first series of this problem, the convergent series:

\[ A = \sum_{n=1}^{\infty} \frac{n4^n}{32n}. \]

If we write out the terms of this series, we get:

\[
A = \frac{4}{9} + 2 \left( \frac{4}{9} \right)^2 + 3 \left( \frac{4}{9} \right)^3 + 4 \left( \frac{4}{9} \right)^4 + \ldots
\]

\[
= \frac{4}{9} \left( 1 + 2 \left( \frac{4}{9} \right) + 3 \left( \frac{4}{9} \right)^2 + 4 \left( \frac{4}{9} \right)^3 + \ldots \right)
\]

\[
= \frac{4}{9} \cdot \frac{G(\frac{4}{9})}{9}
\]

\[
= \frac{4}{9} \cdot \frac{1}{(1 - \frac{4}{9})^2}
\]

\[
= \frac{4}{9} \left( \frac{81}{(9 - 4)^2} \right)
\]

\[
= \frac{36}{25} = 1.44.
\]
Question 3

Solve each of the following differential equations and discuss the behavior of each solution:

- \( \frac{dy}{dt} + 3y = 2 \cos(2t) \), \( y(0) = 4 \).

We first find a particular solution.

We try \( y = A \cos(2t) + B \sin(2t) \).

Then we need:

\[
2 \cos(2t) = y' + 3y = -2A \sin(2t) + 2B \cos(2t) + 3(A \cos(2t) + B \sin(2t)),
\]

\[
0 = \sin(2t)(-2A + 3B) + \cos(2t)(2B + 3A - 2).
\]

So we want \(-2A + 3B = 0\) and \(2B + 3A - 2 = 0\).

The first equation gives \(-6A + 9B = 0\), the second \(4B + 6A - 4 = 0\).

Adding these equations gives: \(13B - 4 = 0\), so \(B = \frac{4}{13}\).

Then \(A = \frac{3B}{2} = \frac{6}{13}\).

So we have the particular solution:

\[
y_p = \frac{2}{13}(3 \cos(2t) + 2 \sin(2t)).
\]

The associated homogeneous problem is:

\[
\frac{dy}{dt} + 3y = 0,
\]

\[
\frac{dy}{dt} = -3y.
\]

We recognize this as a special case of the standard equation for exponential growth/decay, with general solution:

\[
y_h = Ce^{-3t}.
\]

So the general solution of the given problem is:

\[
y = y_h + y_p = Ce^{-3t} + \frac{2}{13}(3 \cos(2t) + 2 \sin(2t)).
\]
For the initial condition, \( y(0) = 4 \), we put \( t = 0 \) and \( y = 4 \), giving:

\[
4 = C + \frac{6}{13},
\]

\[
C = \frac{46}{13}.
\]

So the required solution is:

\[
y = \frac{2}{13} (23e^{-3t} + 3 \cos(2t) + 2 \sin(2t)).
\]

Plotting, we see that for \( t \) negative the graph is always concave up and decreasing and \( y \) goes to infinity as \( t \to -\infty \).

For \( t \) positive, the exponential term quickly dies off and the solution is then an almost perfect sinusoidal curve, of amplitude \( \frac{2}{\sqrt{13}} = 0.55470 \) and period \( \pi \).

Indeed already by the first local maximum, which occurs at \( t = 3.43543 \), gives the \( y \)-value \( y = 0.55482 \), which is very close to the sinusoidal amplitude.
\[ \frac{dy}{dt} = \frac{y^3}{ty + 2t}, \quad y(1) = 2. \]

This is separable, we separate and integrate:

\[
\frac{dy}{dt} = \frac{y^3}{ty + 2t} = \frac{y^3}{t(y + 2)},
\]

\[
\frac{(y + 2)dy}{y^3} = \frac{dt}{t},
\]

\[
\int \frac{(y + 2)dy}{y^3} = \int \frac{dt}{t},
\]

\[
\ln(|t|) = \int \left( \frac{1}{y^2} + \frac{2}{y^3} \right) dy = C - y^{-1} - y^{-2},
\]

\[
\ln(|t|) + y^{-1} + y^{-2} = C.
\]

The initial condition \( y(1) = 2 \) gives:

\[
C = \ln(1) + \frac{1}{2} + \frac{1}{4} = \frac{3}{4},
\]

\[
0 = (\ln(|t|) - \frac{3}{4}) + y^{-1} + y^{-2}.
\]

\[
y^2 \left( \ln(|t|) - \frac{3}{4} \right) + y + 1 = 0,
\]

\[
y = \frac{1}{2 \left( \ln(t) - \frac{3}{4} \right)} \left( -1 \pm \sqrt{1 - 4 \left( \ln(|t|) - \frac{3}{4} \right)} \right)
\]

\[
= \frac{2}{(4 \ln(|t|) - 3)} \left( -1 \pm 2 \sqrt{1 - \ln(|t|)} \right)
\]

When \( t = 1 \), \( y \) is positive, which requires the negative square root, giving the required solution as:

\[
y = \frac{2}{3 - 4 \ln(|t|)} \left( 1 + 2 \sqrt{1 - \ln(|t|)} \right).
\]
Plotting the graph of the solution we see that it is defined for the interval $|t| < e^{\frac{2}{3}} = 2.117000017$.

The graph is symmetrical about the $t$-axis.

For positive $t$ it has a cusp of infinite slope at the origin.

Since the slope is infinite there, even though $y$ is well-defined, we should probably regard the solution as being strictly valid for $t > 0$ only, so on the open interval $\left(0, e^{\frac{2}{3}}\right)$.

Then $y$ increases steadily from 0, going to infinity as $t \to \left(e^{\frac{2}{3}}\right)^-$. The graph is initially concave down and switches to concave up at the inflection point: $(0.3538581501, 1.077718305)$.

The $y$-value of the inflection point is the unique positive root $y_+$ of the cubic $2y^3 + 5y^2 - 4y - 4 = 0$, then the $t$-value is $e^{\frac{2}{3} - \frac{1}{y_+} - \frac{1}{y_+^2}}$.

We may give an explicit formula for $y_+$:

$$y_+ = \frac{1}{6}(-5 + 7\cos(\theta) + 7\sqrt{3}\sin(\theta)),$$

Here $\theta$ is given as follows: $3\theta = \arctan\left(\frac{12}{89}\sqrt{762}\right)$.

The cubic arises by taking the logarithmic derivative of the differential equation:

$$y' = \frac{y^3}{t(y + 2)}, \quad \ln(y') = 3\ln(y) - \ln(y + 2) - \ln(t),$$

$$y'' = \frac{3y'}{y} - \frac{y'}{y + 2} - \frac{1}{t}$$

$$= \frac{1}{t} \left(\frac{3y^2}{(y + 2)} - \frac{y^3}{(y + 2)^2} - 1\right)$$

$$= \frac{1}{t(y + 2)^2}(3y^2(y + 2) - y^3 - (y + 2)^2)$$

$$= \frac{1}{t(y + 2)^2}(2y^3 + 5y^2 - 4y - 4),$$

$$y'' = \frac{y^3}{t^2(y + 2)^3}(2y^3 + 5y^2 - 4y - 4).$$

So the inflection point arises, for $y$ given by the positive root of the cubic, as described above.