Honors Calculus  Quiz 6 Solutions  10/21/5

Question 1

A 50 foot ladder has its upper end on a vertical wall and its lower end on a horizontal floor.
Its lower end is slipping away from the wall at a constant rate of 1 foot per second.
How fast is its upper end moving when the lower end is 30 feet away from the wall?
As the ladder falls does the upper end accelerate or decelerate? Explain.

Let the upper end of the ladder be at the point \((0, y)\) and the lower end at \((x, 0)\),
where the origin is at the foot of the wall and \(x\) and \(y\) are non-negative.
Then we have, by Pythagoras’:

\[
x^2 + y^2 = 50^2 = 2500.
\]

Differentiating this with respect to time, we get:

\[
2xx' + 2yy' = 0,
\]

\[
yy' = -xx',
\]

\[
y' = -\frac{xx'}{y}.
\]

We are given that \(x' = 1\).
When \(x = 30\), we have \(y^2 + 30^2 = 2500\), so \(y^2 = 2500 - 900 = 1600 = 40^2\), so we have \(y = 40\).
Substituting, we get:

\[
y' = -\frac{30(1)}{40} = -\frac{3}{4}.
\]
To find the acceleration, we differentiate the relation \( y' = -\frac{xx'}{y} \) with respect to \( t \), using the quotient rule, giving:

\[
y'' = \frac{- (x')^2 y - xx'' y - (-xx')y'}{y^2},
\]

\[
= \frac{-(x')^2 y^2 - xx'' y^2 - (-xx')yy'}{y^3},
\]

\[
= \frac{-(x')^2 y^2 - xx'' y^2 - (-xx')^2}{y^3},
\]

\[
= \frac{-(x')^2(x^2 + y^2) - xx'' y^2}{y^3},
\]

\[
= \frac{-2500(x')^2 - xx'' y^2}{y^3},
\]

We are given that \( x' = 1 \), a constant, so \( x'' = 0 \).

Substituting, we get:

\[
y'' = -\frac{2500}{y^3}.
\]

So the acceleration is negative and goes to infinity as the top of the ladder hits the ground.

At the time when \( x = 30 \), we have in units of feet per second per second:

\[
y'' = \frac{-2500}{40^3} = \frac{-2500}{64000} = -\frac{25}{640} = -\frac{5}{128}.
\]

Alternatively, since \( x' = 1 \), we may write \( x = 30 + t \), starting time at the moment that \( x = 30 \).

Then we have:

\[
y^2 = 2500 - x^2 = 2500 - (30 + t)^2 = 1600 - 60t - t^2,
\]

\[
y = \sqrt{1600 - 60t - t^2},
\]

\[
y' = \frac{1}{2\sqrt{1600 - 60t - t^2}}(60 - 2t) = -\frac{30 + t}{\sqrt{1600 - 60t - t^2}}.
\]

We evaluate when \( x = 30 \), so when \( t = 0 \), giving the rate of change of \( y \) in feet per second:

\[
y' = \frac{-30}{\sqrt{1600}} = -\frac{3}{4}.
\]
For the acceleration, we note that $y' = -\frac{x}{y}$.

The quantity $x$ is positive and increasing, $y$ is positive and decreasing, so $y'$ is decreasing, so the acceleration is negative and goes to minus infinity as $y$ goes to zero, which occurs at time $t = 20$ seconds.

Alternatively we differentiate again, using the quotient rule:

$$y'' = -\frac{1(\sqrt{1600 - 60t - t^2}) - (30 + t) \left( \frac{1}{2\sqrt{1600 - 60t - t^2}} \right)(-60 - 2t)}{1600 - 60t - t^2}$$

$$= -\left(\sqrt{1600 - 60t - t^2}\right)^{-\frac{3}{2}} (1600 - 60t - t^2 + (30 + t)^2)$$

$$= -\left(\sqrt{1600 - 60t - t^2}\right)^{-\frac{3}{2}} (1600 - 60t - t^2 + 900 + 60t + t^2)$$

$$= -2500(\sqrt{1600 - 60t - t^2})^{-\frac{3}{2}}.$$

So $y'' < 0$ and the acceleration is negative.
Question 2

Let \( f(x) = x^3 - 3x^2 - 9x + 20 \), defined for any real \( x \).

- Sketch the graph of the function \( f \).

We have:

\[
\begin{align*}
- f'(x) &= 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1), \\
- f(-1) &= -1 - 3 + 9 + 20 = 25, \\
- f(3) &= 27 - 27 - 27 + 20 = -7, \\
- f''(x) &= 6x - 6 = 6(x - 1), \\
- f(1) &= 1 - 3 - 9 + 20 = 9.
\end{align*}
\]

This is then a standard cubic curve concave down for \( x < 1 \) and concave up for \( x > 1 \), with an inflection point at \((1,9)\).

It increases until reaching a local maximum at \((-1,25)\) then decreases to its local minimum at \((3,-7)\) and then increases.

As \( x \to -\infty \), we have \( f(x) \to -\infty \).

As \( x \to \infty \), we have \( f(x) \to \infty \).

The \( y \)-intercept is at \((0,20)\).

Plotting the graph, we notice that there appears to be an \( x \)-intercept at \( x = 4 \).

We evaluate \( f(4) = 4^3 - 3(4)^2 - 9(4) + 20 = 64 - 48 - 36 + 20 = 0 \).

So \( f(x) \) has a factor of \( x - 4 \).

Factoring, we get:

\[
x^3 - 3x^2 - 9x + 20 = (x - 4)(x^2 + x - 5).
\]

So by the quadratic formula, the other \( x \)-intercepts are at \( x = \frac{1}{2} ( -1 \pm \sqrt{21} ) \).
• Prove that the function $f$ has a unique zero in the interval $[0, 3]$. The function $f$ is a continuous function, since it is a polynomial. We have $f(0)f(3) = 20(-7) < 0$, so $f$ changes sign on the interval $[0, 3]$, so has a root in that interval, by the Intermediate Value Theorem. The root in question is unique, because $f'(x) = 3(x - 3)(x + 1) < 0$ in the interval $(0, 3)$, so $f$ is strictly decreasing in the interval $(0, 3)$, so is one-to-one in that interval, so takes any value at most once.

• Using Newton’s method with at least three iterations for a suitable starting value, estimate the position of the zero.

Newton’s method maps:

$$x \rightarrow x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - 3x^2 - 9x + 20}{3x^2 - 6x - 9}$$

$$= \frac{x(3x^2 - 6x - 9) - x^3 + 3x^2 + 9x - 20}{3x^2 - 6x - 9}$$

$$= \frac{3x^3 - 6x^2 - 9x - x^3 + 3x^2 + 9x - 20}{3x^2 - 6x - 9}$$

$$= \frac{2x^3 - 3x^2 - 20}{3x^2 - 6x - 9}.$$

If we start with initial value $x = 2$, Maple gives the iterates:

$$2, \ 2.16, \ 2.532, \ 2.474993808, \ 2.48294331315812477075581884.$$  

As decimals these are:

$$2, \ 1.7, \ 1.791245, \ 1.791287847063115226, \ 1.791287847477920003.$$

If we start from $x = 1.5$ we get the identical sequence after the first term. If we start from $x = 1$, we get the same decimal result after five iterations.

• By factoring the function $f$, or otherwise, compare your results with the exact value for the zero.

Maple gives the decimal value of the root $\frac{1}{2} \left( -1 + \sqrt{21} \right)$ as $1.791287847477920003$ in exact agreement with our estimate to eighteen decimal places.
Question 3

A basket is strung between two (vertical) trees on horizontal ground. The trees are 20 meters apart.
On rope is attached to a tree at a height of 20 meters.
The other rope is attached to the other tree at a height of 25 meters.
The basket is 10 meters above the ground and is vertically above a point half-way between the two trees.
If the basket has a mass of 80 kilograms, find the forces in the ropes (assuming that the ropes are straight and of negligible thickness and neglecting the weight of the ropes themselves).

Take the origin $O = (0,0)$ at the basket, with the $y$-axis vertical.
Then one rope is attached at $A = [-10, 10]$ and the other at $B = [10, 15]$.

- The force $\mathbf{F}$ in the rope $OA$ is then:
  \[ \mathbf{F} = s \mathbf{OA} = s[-10, 10], \]for some scalar $s$.

- The force $\mathbf{H}$ in the rope $OB$ is then:
  \[ \mathbf{H} = t \mathbf{OB} = t[10, 15], \]for some scalar $t$.

- The gravitational force on the basket of mass $m = 80$ in the gravitational field $\mathbf{G} = [0, -g]$ is: $m \mathbf{G} = [0, -80g]$.

The force balance equation is then:
\[ 0 = \mathbf{F} + \mathbf{H} + m \mathbf{G}, \]
\[ [0, 0] = s[-10, 10] + t[10, 15] + [0, -80g], \]
\[ -10s + 10t = 0, \quad 0 = 10s + 15t - 80g, \]
\[ s = t, \quad 0 = 25t - 80g, \quad s = t = \frac{80g}{25} = \frac{16g}{5}. \]

Then the forces are:
\[ \mathbf{F} = \frac{16g}{5}[-10, 10] = [-32g, 32g] = 32g[-1, 1], \]
\[ \mathbf{H} = \frac{16g}{5}[10, 15] = [32g, 48g] = 16g[2, 3]. \]
It is clear that the three forces on the basket add to zero, as required.
The force $F$ has size $32g\sqrt{2}$.
The force $H$ has size $16g\sqrt{13}$.
If we take $g = 9.8$, the forces in Newtons are:

$$F = [-313.6, 313.6], \quad |F| = 443.5,$$
$$H = [313.6, 470.4], \quad |H| = 565.4.$$  

Alternatively we resolve the forces in the vertical and horizontal directions.
The angle that $F$ makes with the horizontal is 45 degrees, or 135 degrees to the positive $x$-axis.
The angle that $H$ makes with the positive $x$-axis is:

$$\theta = \arcsin \left( \frac{15}{\sqrt{10^2 + 15^2}} \right) = \arcsin \left( \frac{3}{\sqrt{13}} \right) = \arccos \left( \frac{10}{\sqrt{10^2 + 15^2}} \right) = \arccos \left( \frac{2}{\sqrt{13}} \right).$$

Then the horizontal force balance equation is:

$$|F| \cos \left( \frac{3\pi}{4} \right) + |H| \cos(\theta) = 0, \quad -|F| \frac{1}{\sqrt{2}} + |H| \frac{2}{\sqrt{13}} = 0.$$  

Then the vertical force balance equation is:

$$|F| \sin \left( \frac{\pi}{4} \right) + |H| \sin(\theta) = 80g, \quad |F| \frac{1}{\sqrt{2}} + |H| \frac{3}{\sqrt{13}} = 80g.$$  

Adding the two equations eliminates $|F|$ and gives:

$$|H| \frac{5}{\sqrt{13}} = 80g, \quad |H| = 16g\sqrt{13}.$$  

Back-substituting gives:

$$|F| \frac{1}{\sqrt{2}} = |H| \frac{2\sqrt{2}}{\sqrt{13}} = 16g\sqrt{13} \left( \frac{2\sqrt{2}}{\sqrt{13}} \right) = 32g\sqrt{2}.$$  

Then we have:

$$F = |F| \begin{bmatrix} \cos \left( \frac{3\pi}{4} \right), \sin \left( \frac{3\pi}{4} \right) \end{bmatrix} = 32g\sqrt{2} \begin{bmatrix} -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \end{bmatrix} = [-32g, 32g],$$

$$H = |H| \begin{bmatrix} \cos(\theta), \sin(\theta) \end{bmatrix} = 16g\sqrt{13} \begin{bmatrix} \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \end{bmatrix} = [32g, 48g].$$

These results agree with those obtained earlier.
Another method is to resolve the forces in the direction perpendicular to one of them: this immediately gives us the size of the other force.

- If we resolve perpendicular to \( \mathbf{H} \), we get the equation:

\[
|\mathbf{F}| \cos \left( \theta - \frac{\pi}{4} \right) = 80g \cos(\theta),
\]

\[
|\mathbf{F}| = \frac{80g \cos(\theta)}{\cos \left( \theta - \frac{\pi}{4} \right)}
= \frac{80g \cos(\theta)}{\cos(\theta) \cos \left( \frac{\pi}{4} \right) + \sin(\theta) \sin \left( \frac{\pi}{4} \right)}
= \frac{80g \sqrt{2}}{1 + \tan(\theta)}
= \frac{80g \sqrt{2}}{1 + \frac{1}{\sqrt{2}}}
= \frac{80g \sqrt{2}}{\frac{5}{2}} = 32g \sqrt{2}.
\]

- If we resolve perpendicular to \( \mathbf{F} \), we get the equation:

\[
|\mathbf{H}| \cos \left( \theta - \frac{\pi}{4} \right) = 80g \cos \left( \frac{\pi}{4} \right),
\]

\[
|\mathbf{H}| = \frac{80g \cos \left( \frac{\pi}{4} \right)}{\cos \left( \theta - \frac{\pi}{4} \right)}
= \frac{80g \cos \left( \frac{\pi}{4} \right)}{\cos(\theta) \cos \left( \frac{\pi}{4} \right) + \sin(\theta) \sin \left( \frac{\pi}{4} \right)}
= \frac{80g}{\cos(\theta) + \sin(\theta)}
= \frac{80g}{\frac{2}{\sqrt{13}} + \frac{3}{\sqrt{13}}}
= \frac{80g}{\frac{5}{\sqrt{13}}} = 16g \sqrt{13}.
\]

Again our results agree with those obtained earlier.