Honors Calculus I Quiz 4 Solutions 9/23/5

Question 1

Let \( f(x) = x^2 + 4x - 5 \).
Show using the definition of the limit that the following limit is true:
\[
\lim_{x \to 2} f(x) = 7.
\]
Also determine an open interval \( J \) containing the point \( x = 2 \), such that for any \( x \) in \( J \) we have \( |f(x) - 7| < 0.5 \).

Given \( \epsilon > 0 \), we need an open interval \( J_\epsilon \) such that \( 2 \in J_\epsilon \) and if \( x \in J_\epsilon \), with \( x \neq 2 \), then \( |f(x) - 7| < \epsilon \).
So we solve the inequality:
\[
|f(x) - 7| < \epsilon,
\]
\[
-\epsilon < x^2 + 4x - 5 - 7 < \epsilon,
\]
\[
12 - \epsilon < x^2 + 4x < 12 + \epsilon,
\]
\[
12 - \epsilon < x^2 + 4x + 4 - 4 < 12 + \epsilon,
\]
\[
12 - \epsilon < (x + 2)^2 - 4 < 12 + \epsilon,
\]
\[
16 - \epsilon < (x + 2)^2 < 16 + \epsilon.
\]
Without loss of generality, we may assume that \( \epsilon < 16 \) and \( x + 2 > 0 \), so we can take the square root to give:
\[
\sqrt{16 - \epsilon} < x + 2 < \sqrt{16 + \epsilon},
\]
\[
\sqrt{16 - \epsilon} - 2 < x < \sqrt{16 + \epsilon} - 2.
\]
Put \( J_\epsilon = (\sqrt{16 - \epsilon} - 2, \sqrt{16 + \epsilon} - 2) \).
Then \( J_\epsilon \) is an open interval.
Also \( \sqrt{16 - \epsilon} - 2 < \sqrt{16 - 2} = 4 - 2 = 2 \) and \( \sqrt{16 + \epsilon} - 2 > \sqrt{16 - 2} = 4 - 2 = 2 \),
so \( 2 \in J_\epsilon \) and for any \( x \in J_\epsilon \), we have \( |f(x) - 7| < \epsilon \), as required, so we are done.

In the particular case that \( \epsilon = 0.5 \), we may take:
\[
J = J_{0.5} = (\sqrt{16 - 0.5} - 2, \sqrt{16 + 0.5} - 2) = (\sqrt{15.5} - 2, \sqrt{16.5} + 2) = (1.937004, 2.062019).
\]
Question 2

Let a differentiable function $f(t)$ satisfy the properties: $f'(t) > 0$, $f(2) = 4$ and $f'(2) = 3$.

- Estimate the quantity $f(2.5)$.

We use the linear approximation based at $x = 2$:

$$y = f(2) + f'(2)(t - 2) = 4 + 3(t - 2) = 3t - 2.$$  

Putting $t = 2.5$ gives our estimate for $f(2.5)$ as: 
$3(2.5) - 2 = 7.5 - 2 = 5.5$.

- Explain why the inverse function $f^{-1}(t)$ exists and find the equation of the tangent line to the graph of the function $y = f^{-1}(t)$ at $t = 4$.

The inverse function exists on any domain consisting of an open interval, because $f'(t) > 0$, so $f$ is increasing, so is one-to-one and invertible as a map from its domain to the image of that domain.

We have $f(2) = 4$, so $f^{-1}(4) = 2$. 
In general we have, if $f(a) = b$ (and $f^{-1}(b) = a$):

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$  

So here we have:

$$(f^{-1})'(4) = \frac{1}{f'(2)} = \frac{1}{3}.$$  

So the required tangent line has the equation:

$$y = f^{-1}(4) + (f^{-1})'(4)(t - 4) = 2 + \frac{1}{3}(t - 4) = \frac{1}{3}(t + 2).$$  

Alternatively, since $f(2) = 4$, the equation of the tangent line to $y = f^{-1}(t)$ at $t = 4$ is simply the inverse function of the tangent line equation to $y = f(t)$ at $t = 2$, which we have found above to be $y = 3t - 2$. 
Interchanging $y$ and $t$ this becomes: $t = 3y - 2$, or $3y = t + 2$, so the required equation is $y = \frac{1}{3}(t + 2)$, as before.
**Question 3**

Let \( f(x) = x^3 - 8x^2 + 5x + 14. \)

- Find the intervals on which \( f'(x) > 0 \) and the intervals on which \( f'(x) < 0. \)

We have \( f'(x) = 3x^2 - 16x + 5 = (3x - 1)(x - 5) \) and \( f''(x) = 6x - 16. \) Note that \( f''(x) = 0 \) when \( x = \frac{16}{6} = \frac{8}{3}. \)

- If \( x = 5, \) then \( f'(x) = 0. \)
- If \( x = \frac{1}{3}, \) then \( f'(x) = 0. \)
- If \( x > 5, \) then \( f'(x) > 0, \) since both \( x - 5 \) and \( 3x - 1 \) are positive.
- If \( x < \frac{1}{3}, \) then \( f'(x) > 0, \) since both \( x - 5 \) and \( 3x - 10 \) are negative.
- If \( \frac{1}{3} < x < 5, \) then \( f'(x) < 0, \) since \( x - 5 < 0 \) and \( 3x - 1 > 0. \)

Since we have considered all possible values for \( x, \) we see that:

- \( f'(x) > 0 \) on \((-\infty, \frac{1}{3}) \cup (5, \infty). \)
- \( f'(x) < 0 \) on \( (\frac{1}{3}, 5). \)

- Sketch the graphs of the functions \( y = f(x), \ y = f'(x) \) and \( y = f''(x) \) on the same sketch.

We have \( f(-1) = (-1)^3 - 8(-1)^2 + 5(-1) + 14 = -1 - 8 - 5 + 14 = 0, \) so \( f(x) \) has a factor of \( x + 1. \)

We factor \( f(x) \) completely:

\[
f(x) = x^3 - 8x^2 + 5x + 14 = (x + 1)(x^2 - 9x + 14)
\]

\[
= (x + 1)(x - 2)(x - 7).
\]
Note that we have:

- \( f(-1) = f(2) = f(7) = 0 \),
- \( f(0) = 14 \),
- \( f(5) = (5 + 1)(5 - 2)(5 - 7) = 6(3)(-2) = -36 \).

\[
- f \left( \frac{1}{3} \right) = \left( \frac{1}{3} + 1 \right) \left( \frac{1}{3} - 2 \right) \left( \frac{1}{3} - 7 \right)
= \frac{1}{3^3}(1 + 3)(1 - 6)(1 - 21) = \frac{1}{27}(4)(-5)(-20) = \frac{400}{27} = 14.814
\]

\[
- f \left( \frac{8}{3} \right) = \left( \frac{8}{3} + 1 \right) \left( \frac{8}{3} - 2 \right) \left( \frac{8}{3} - 7 \right)
= \frac{1}{3^3}(8 + 3)(8 - 6)(8 - 21) = \frac{1}{27}(11)(2)(-13) = -\frac{286}{27} = -10.592
\]

The graph of \( f(x) \) is then a standard cubic curve, increasing from \(-\infty\) as \( x \to -\infty \), crossing the \( x \)-axis first at \((-1, 0)\), crossing the \( y \)-axis at \((0, 14)\) and rising to its local maximum at \( \left( \frac{1}{3}, \frac{400}{27} \right) \), then decreasing, crossing the \( x \)-axis again at \((2, 0)\), going down to its local minimum at \((5, -36)\), then increasing, crossing the \( x \)-axis for the last time at \((7, 0)\) and increasing to \( \infty \) as \( x \to \infty \).

The graph is concave down for \( x < \frac{8}{3} \), inflects at \( \left( \frac{8}{3}, \frac{-286}{27} \right) \) and is concave up for \( x > \frac{8}{3} \).

The graph of \( f'(x) = (3x - 1)(x - 5) \) is then a standard parabola, opening upwards, crossing the \( x \)-axis at \( \left( \frac{1}{3}, 0 \right) \) and at \((5, 0)\) and the \( y \)-axis at \((0, 5)\).

It is symmetrical about the line \( x = \frac{8}{3} \) and its vertex and absolute minimum is at \( \left( \frac{8}{3}, \frac{-49}{3} \right) \).

The graph of \( f''(x) = 6x - 16 \) is a straight line of slope 6, crossing the \( x \)-axis at \( \left( \frac{8}{3}, 0 \right) \) and the \( y \)-axis at \((0, -16)\).
• Which is the largest open interval \( J \), containing the point \( x = 2 \), on which \( f \) has an inverse \( f^{-1} \)?

Explain your answer and give the domain \( K \) of the function \( f^{-1} \).

Also sketch the graph of the function \( f^{-1} \) for the domain \( K \).

\( f(x) \) is strictly decreasing on the interval \( \left( \frac{1}{3}, 5 \right) \) and strictly increasing for \( x \leq \frac{1}{3} \) and for \( x \geq 5 \), so the largest open interval \( J \), containing the point \( x = 2 \), on which \( f \) has an inverse \( f^{-1} \) is \( J = \left( \frac{1}{3}, 5 \right) \): \( f(x) \) fails

the horizontal line test if we go below \( x = \frac{1}{3} \) or above \( x = 5 \).

Then since \( f \) is decreasing and continuous on \( J \), we have:

\[
K = \left( f(5), f\left( \frac{1}{3} \right) \right) = \left( -36, \frac{400}{27} \right).
\]

The graph of \( y = f^{-1}(x) \) is the reflection in the line \( y = x \) of the graph of \( f(x) \) for the domain \( J \).

\( f^{-1} \) has domain \( K \), range \( J \).

Its graph is S-shaped and everywhere decreasing with its inflection point at \( \left( -\frac{49}{3}, \frac{8}{3} \right) \).

It is concave down for \( x < -\frac{49}{3} \) and concave up for \( x > \frac{49}{3} \).
Find the equation of the tangent line to the graph of the function \( y = f^{-1}(x) \) at the point with \( x = 0 \).

For \( x \in \mathcal{J} \), we have \( f(x) = 0 \) iff \( x = 2 \).
So \( f^{-1}(0) = 2 \).
Note that \( f(2) = 0 \) and \( f'(2) = (3(2) - 1)(2 - 5) = (5)(-3) = -15 \).
The tangent line to \( y = f(x) \) at \( x = 2 \) is:

\[
y = f(2) + f'(2)(x - 2) = 0 - 15(x - 2).
\]

Interchanging \( x \) and \( y \), and solving for \( y \), we get the required equation of the tangent line to \( y = f^{-1}(x) \) at \( x = 0 \):

\[
x = -15(y - 2),
\]

\[
y - 2 = -\frac{x}{15},
\]

\[
y = -\frac{x}{15} + 2 = \frac{1}{15}(30 - x).
\]

Alternatively, we have, since \( f(0) = 2 \):

\[
(f^{-1})'(0) = \frac{1}{f'(2)} = \frac{1}{-15} = -\frac{1}{15}.
\]

Then the required equation is:

\[
y = f(0) + f'(0)(x - 0) = 2 - \frac{1}{15}x = \frac{1}{15}(30 - x).
\]