Honors Calculus I Quiz 3 Solutions 9/16/5

Question 1
Let $f(x) = \frac{1}{x^2}$ and $g(x) = x^3 + 1$.

- What are the domains and ranges of $f$ and $g$?
  
  - The function $f$ is defined for $x \neq 0$ and has range $\mathbb{R}^+$, the set of all positive real numbers, since its outputs are clearly strictly positive real numbers and since given any $y > 0$, we have:
    
    $$f\left(\frac{1}{\sqrt{y}}\right) = \frac{1}{\left(\frac{1}{\sqrt{y}}\right)^2} = \frac{1}{\frac{1}{y}} = y.$$  

    So $f$ gives a surjective map $f : \mathbb{R} - \{0\} \to \mathbb{R}^+$.
  
  - The function $g$ is defined for all real $x$ and has range $\mathbb{R}$, the set of all real numbers, since given any real number $y$, we have $g(\sqrt[3]{y - 1}) = (\sqrt[3]{y - 1})^3 + 1 = y - 1 + 1 = y$, so $g$ gives a surjective map $g : \mathbb{R} \to \mathbb{R}$.

- One of these maps is invertible and one is not. Which is which and why?
  
  - The map $f$ is not invertible, since $f(1) = f(-1) = 1$, so the inputs $-1$ and $1$ both give the same output and $f$ is not one-to-one.
  
  - The map $g$ is invertible since the equation $y = x^3 + 1$ has the unique real solutions for $x$ given $y$:
    
    $$x = \sqrt[3]{y - 1}.$$  

    So the inverse function is $g^{-1}(x) = \sqrt[3]{x - 1}$.

Check:

Both the functions $g$ and $g^{-1}$ are well defined for all real inputs and we have, for any real number $x$:

$$g(g^{-1}(x)) = g(\sqrt[3]{x - 1}) = (\sqrt[3]{x - 1})^3 + 1 = x - 1 + 1 = x,$$

$$g^{-1}(g(x)) = \sqrt[3]{(g(x) - 1)} = \sqrt[3]{x^3 + 1 - 1} = \sqrt[3]{x^3} = x.$$
• Give formulas for the compositions \( f \circ f, f \circ g, g \circ g \) and \( g \circ f \) and obtain the derivatives of each of those compositions. Note that we have \( f'(x) = -2x^{-3} \) and \( g'(x) = 3x^2 \).

- We have, for any real \( x \neq 0 \):
  \[
  (f \circ f)(x) = f(f(x)) = \frac{1}{(f(x))^2} = \frac{1}{\left(\frac{1}{x^4}\right)^2} = \frac{1}{x^8} = x^4,
  \]
  \[
  (f \circ f)'(x) = 4x^3.
  \]
  Alternatively we can use the chain rule to determine \( (f \circ f)'(x) \):
  \[
  (f \circ f)'(x) = f'(f(x))f'(x) = \left(\frac{-2}{f(x)^3}\right)\left(-\frac{2}{x^3}\right) = \frac{-2}{x^3}\left(-\frac{2}{x^3}\right) = -2x^6\left(-\frac{2}{x^3}\right) = 4x^3.
  \]

- We have, for any real \( x \neq -1 \):
  \[
  (f \circ g)(x) = f(g(x)) = \frac{1}{(g(x))^2} = \frac{1}{(x^3 + 1)^2},
  \]
  \[
  (f \circ g)'(x) = -2(x^3 + 1)^{-3}(3x^2) = -6x^2(x^3 + 1)^{-3}.
  \]
  Alternatively we can use the chain rule to determine \( (f \circ g)'(x) \):
  \[
  (f \circ g)'(x) = f'(g(x))g'(x) = \left(\frac{-2}{(g(x))^3}\right)3x^2 = -2(x^3+1)^{-3}(3x^2) = -6x^2(x^3+1)^{-3}.
  \]

- We have, for any real \( x \neq 0 \):
  \[
  (g \circ f)(x) = g(f(x)) = (f(x))^3 + 1 = (x^{-2})^3 + 1 = x^{-6} + 1,
  \]
  \[
  (g \circ f)'(x) = -6x^{-7}.
  \]
  Alternatively we can use the chain rule to determine \( (g \circ f)'(x) \):
  \[
  (g \circ f)'(x) = g'(f(x))f'(x) = 3(f(x))^2\left(-\frac{2}{x^3}\right) = 3x^{-4}(-2x^{-3}) = -6x^{-7}.
  \]

- We have, for any real \( x \):
  \[
  (g \circ g)(x) = g(g(x)) = (g(x))^3 + 1 = (x^3 + 1)^3 + 1,
  \]
  \[
  (g \circ g)'(x) = 3(x^3 + 1)^2(3x^2) = 9x^2(x^3 + 1)^2.
  \]
  Alternatively we can use the chain rule to determine \( (g \circ g)'(x) \):
  \[
  (g \circ g)'(x) = g'(g(x))g'(x) = 3(g(x))^2(3x^2) = 3(x^3+1)^2(3x^2) = 9x^2(x^3+1)^2.
  \]
**Question 2**

Let \( f'(t) = g(t) \) and \( g'(t) = -f(t) \).

Prove that \( f^2 + g^2 = C \) is constant.

If \( f(0) = 0 \) and \( f'(0) = 1 \), evaluate the constant \( C \).

Also compute the derivative of the function \( h(t) = \frac{f(t)}{g(t)} \).

We have:

\[
\frac{d}{dt}(f^2(t) + g^2(t)) = 2f(t)f'(t) + 2g(t)g'(t) = 2f(t)g(t) + 2g(t)(-f(t)) = 0.
\]

So the function \( f^2 + g^2 \) has zero derivative, so is constant on any open interval of its definition.

We have \( C = f^2(0) + g^2(0) = f^2(0) + (f'(0))^2 \).

Putting \( t = 0 \) gives \( C = f^2(0) + (f'(0))^2 = 0^2 + 1^2 = 1 \).

Finally we have, by the division rule:

\[
h' = \left( \frac{f}{g} \right)' = \frac{f'g - g'f}{g^2} = \frac{g(g) - (-f)f}{g^2} = \frac{g^2 + f^2}{g^2} = \frac{1}{g^2}.
\]

The model here is \( f = \sin(t) \) and \( g = \cos(t) \), since we have the required relations:

- \( f'(t) = \sin'(t) = \cos(t) = g(t) \),
- \( g'(t) = \cos'(t) = -\sin(t) = -f(t) \),
- \( f(0) = \sin(0) = 0 \) and \( f'(0) = g(0) = \cos(0) = 1 \).

Then the calculation above proves the trigonometric identity:

\[
\sin^2(t) + \cos^2(t) = 1.
\]

Then \( h(t) = \frac{\sin(t)}{\cos(t)} = \tan(t) \) and the calculation of \( h' \) shows that:

\[
h'(t) = \tan'(t) = \frac{1}{g^2(t)} = \frac{1}{\cos^2(t)} = \sec^2(t).
\]
Question 3

Find the linear approximation to the function \( f(x) = (1 + 2x)^{-\frac{2}{3}} \) based at the origin and use it estimate the value of \( f(0.1) = (1.2)^{-\frac{2}{3}} \).

By sketching the graphs of the function \( f(x) \) and its linear approximation on the same graph, determine if your estimate is an under-estimate or an over-estimate.

The linear approximation, \( f_1(x) \) based at \( x = a \) to the function \( f(x) \) is given by the formula:

\[
f_1(x) = f(a) + f'(a)(x - a).
\]

Here we have:

- \( a = 0 \),
- \( f(x) = (1 + 2x)^{-\frac{2}{3}}, \quad f(a) = f(0) = 1 \),
- \( f'(x) = -\frac{2}{3}(1 + 2x)^{-\frac{5}{3}}(2) = -\frac{4}{3}(1 + 2x)^{-\frac{5}{3}}, \quad f'(a) = f'(0) = -\frac{4}{3} \).

So we get:

\[
f_1(x) = f(0) + f'(0)(x - 0) = 1 - \frac{4}{3}x = \frac{1}{3}(3 - 4x).
\]

Putting \( x = 0.1 \), we estimate \((1.2)^{-\frac{2}{3}}\) as:

\[
f_1(0.1) = \frac{1}{3}(3 - 4(0.1)) = \frac{1}{3}(3 - \frac{2}{5}) = \frac{1}{3}\left(\frac{13}{5}\right) = \frac{13}{15} = 0.866\overline{6}.
\]

The graph of the function \( f(x) \) on the interval \([-0.2, 0.2]\) show a gently decreasing curve that is concave up, so the tangent line at the origin lies below the curve, so our estimate using \( f_1(x) \) is an under-estimate.

Alternatively, we compute the second derivative at the origin:

\[
f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\left(-\frac{4}{3}(1 + 2x)^{-\frac{5}{3}}\right)
= -\frac{4}{3}\left(-\frac{5}{3}\right)(1 + 2x)^{-\frac{8}{3}}(2) = \frac{40}{9}(1 + 2x)^{-\frac{8}{3}}, \quad f''(0) = \frac{40}{9} > 0.
\]

Since \( f''(0) \) is positive, the graph is concave up at the origin and the tangent line at the origin lies below the curve, so our estimate is an under-estimate.

Using Maple, we have \((1.2)^{-\frac{2}{3}} = 0.8855488\), so the true value of \((1.2)^{-\frac{2}{3}}\) is higher, as expected; the percentage error in our estimate is 2.132 percent.

Alternatively, we have \((1.2)^{-\frac{2}{3}} > \frac{13}{15} \) iff \((\frac{6}{5})^{-2} > (\frac{13}{15})^3 \) iff 25(15^3) > 36(13^3) iff 25(25)(15) > 4(2097) iff 9375 > 8388, which is true.