Honors Calculus Homework 6 Solutions, due 10/13/5

Question 1

Let \( A = [2, 3], B = [5, 4], C = [-3, -1]. \)

Find all points \( D \) such that the four points \( A, B, C \) and \( D \) form a parallelogram and sketch all the parallelograms on one graph.

What is the area of each such parallelogram?

Where is the centroid of each such parallelogram?

A parallelogram is known if its diagonals are known.

One diagonal uses up two of the three points \( A, B \) and \( C \).

There are three possibilities:

- \( BC \) is a diagonal.
  - Then the other diagonal is \( AD \).
  - The midpoints of \( BC \) and \( AD \) coincide, which gives the equation:
    \[
    \frac{1}{2}(A + D) = \frac{1}{2}(B + C),
    \]
    \[
    \]

- \( CA \) is a diagonal.
  - Then the other diagonal is \( BD \).
  - The midpoints of \( CA \) and \( BD \) coincide, which gives the equation:
    \[
    \frac{1}{2}(B + D) = \frac{1}{2}(C + A),
    \]
    \[
    \]

- \( AB \) is a diagonal.
  - Then the other diagonal is \( CD \).
  - The midpoints of \( AB \) and \( CD \) coincide, which gives the equation:
    \[
    \frac{1}{2}(C + D) = \frac{1}{2}(A + B),
    \]
    \[
    D = A + B - C = [2, 3] + [5, 4] - [-3, -1] = [10, 8].
    \]
The three parallelograms each have twice the area $\Delta$, of the triangle $ABC$. We have:

$$4\Delta^2 = |\mathbf{AB}|^2|\mathbf{AC}|^2 - (\mathbf{AB}, \mathbf{AC})^2$$

$$= |\mathbf{B} - \mathbf{A}|^2|\mathbf{C} - \mathbf{A}|^2 - ((\mathbf{B} - \mathbf{A}), (\mathbf{C} - \mathbf{A}))^2$$

$$= |[3, 1]|^2|[−5, −4]|^2 - ([3, 1], [−5, −4])^2$$

$$= (3^2 + 1^2)((−5)^2 + (−4)^2) - (−15 − 4)$$

$$= 10(41) − 19^2 = 410 − 361 = 49,$$

$$\Delta = \frac{7}{2}.$$

So each parallelogram has area 7 square units.

The three possibilities for $D$ form a triangle which is similar to $ABC$, with edges twice as long and the points $A$, $B$ and $C$ are the midpoints of the sides of the triangle.

The centroid of each parallelogram lies at the intersection of their diagonals, so at the midpoint of one of the edges of the triangle $ABC$. So the three centroids are:

$$\frac{1}{2}(\mathbf{B} + \mathbf{C}) = \frac{1}{2}([5, 4] + [−3, −1]) = \left[1, \frac{3}{2}\right],$$

$$\frac{1}{2}(\mathbf{C} + \mathbf{A}) = \frac{1}{2}([−3, −1] + [2, 3]) = \left[−\frac{1}{2}, 1\right],$$

$$\frac{1}{2}(\mathbf{A} + \mathbf{B}) = \frac{1}{2}([2, 3] + [5, 4]) = \left[\frac{7}{2}, \frac{7}{2}\right].$$
Question 2

A circular disc of radius 2 meters rotates at 2 radians per second in a vertical plane, without slipping, with its lowest point in contact with a straight, horizontal line.

A point $P$, initially in contact with the ground, is marked on the disc.

Find formulas for the position vector $X$ of the point $P$ at time $t$ seconds, its velocity vector $V$ and its acceleration vector $A$.

Find the maximum speed of the motion of $P$.

Also plot the motion.

We choose co-ordinates so that the lowest point of the disc moves along the horizontal $x$-axis, starting from the origin, with the $y$-axis vertically upwards and with the disc rotating clockwise.

- The circumference of the disc moves at a rate of $2(2) = 4$ meters per second, so at time $t$, the base of the disc is at $(4t, 0)$ of the disc, so the center of the disc is at:

  $$C = [4t, 2].$$

- Relative to the center the point $P$ travels in a uniform circular motion. After $t$ seconds, the angle it makes with the downward vertical through the center is $2t$ radians, so its position vector relative to the point $C$ is:

  $$CP = [-2 \sin(2t), -2 \cos(2t)].$$

So the position vector $X$ of the point $P$ at time $t$ is:

$$X = C + CP = [4t, 2] + [-2 \sin(2t), -2 \cos(2t)] = [4t - 2 \sin(2t), 2 - 2 \cos(2t)].$$

Then the velocity vector $V$ is:

$$V = \frac{d}{dt}(X) = [4 - 4 \cos(2t), 4 \sin(2t)].$$

Also the acceleration vector $A$ is:

$$A = \frac{d}{dt}(V) = [8 \sin(2t), 8 \cos(2t)].$$
We have:
\[
\mathbf{V} \cdot \mathbf{V} = (4 - 4 \cos(2t))^2 + (4 \sin(2t))^2 = 16 - 32 \cos(2t) + 16 \cos^2(2t) + 16 \sin^2(2t) \\
= 16 - 32 \cos(2t) + 16 = 32(1 - \cos(2t)) = 64 \sin^2(t).
\]
Here we used the trigonometric identities:
\[
\cos^2(u) + \sin^2(u) = 1, \\
1 - \cos(2u) = 2 \sin^2(u).
\]
Then the speed is:
\[
|\mathbf{V}| = \sqrt{\mathbf{V} \cdot \mathbf{V}} = \sqrt{64 \sin^2(t)} = 8|\sin(t)|.
\]
Since \(\sin(t)\) has maximum and minimum values 1 and \(-1\) respectively, we see that the maximum speed is 8 meters per second, which occurs first at \(t = \frac{\pi}{2}\) seconds and then every \(\pi\) seconds after that time, so at \(t = \frac{\pi}{2}(2n + 1)\), with \(n\) an integer.
At that time the point \(P\) has turned an angle of \(2t = (2n + 1)\pi\), so its maximum speed is attained when \(P\) is at the top of the disc.

If we plot the curve we get a standard curve, called a cycloid, tracing a complete cycle in each interval of time of \(\pi\) seconds.

Note that the slope \(m\) of the velocity vector is:
\[
m = \frac{4 \sin(2t)}{4 - 4 \cos(2t)} = \frac{2 \sin(t) \cos(t)}{2 \sin^2(t)} = \cot(t).
\]
For \(n\) any integer, as \(t \to n\pi^+\) the slope goes to \(\infty\) and as \(t \to n\pi^-\), the slope goes to \(-\infty\).
So going through \(t = n\pi\), the motion switches from going vertically downwards to vertically upwards.
Then the graph has a cusp at \(t = n\pi\), for each integer \(n\).
Question 3

An ant is on the rim of a rotating disc of radius 10 centimeters. If the disc is rotating at 2 radians per second, find the velocity and acceleration of the ant.

If now the ant starts walking radially towards the center of the disc, moving at 2 centimeters per second, relative to the disc, what now is its velocity and acceleration?

Plot the motion of the ant.

We locate the disc at the origin, with the ant initially at \((10, 0)\).

Then its position vector at time \(t\) is:

\[
\mathbf{X} = [10 \cos(2t), 10 \sin(2t)].
\]

Differentiating, we get its velocity as:

\[
\mathbf{V} = \frac{d\mathbf{X}}{dt} = [-20 \sin(2t), 20 \cos(2t)],
\]

\[
\mathbf{A} = \frac{d\mathbf{V}}{dt} = [-40 \cos(2t), -40 \sin(2t)].
\]

For the last part, if we start the ant’s motion inwards at time zero, then its position vector at time \(t\) is multiplied by a factor of \(10 - 2t\), so is now:

\[
\mathbf{Y} = (10 - 2t) \mathbf{X}.
\]

Then its velocity \(\mathbf{W}\) is:

\[
\mathbf{W} = \frac{d\mathbf{Y}}{dt} = (10 - 2t) \mathbf{V} - 2 \mathbf{X}
= [-20 \sin(2t)(10 - 2t) - 20 \cos(2t), 20 \cos(2t)(10 - 2t) - 20 \sin(2t)].
\]

Differentiating again, its acceleration \(\mathbf{B}\) is:

\[
\mathbf{B} = \frac{d\mathbf{W}}{dt} = (10 - 2t) \mathbf{A} - 4 \mathbf{V}
= [-40 \cos(2t)(10 - 2t) + 80 \sin(2t), -40 \cos(2t)(10 - 2t) - 80 \cos(2t)].
\]
Question 4

Find the parametric equations of the line \( AB \) and of its perpendicular bisector.
Also find the two unit vectors that make an angle of 45 degrees with the direction of the line \( AB \).

The direction vector of the line may be taken to be \( \overrightarrow{AB} = B - A = [8, -5] - [2, 3] = [6, -8] \).
If \( \overrightarrow{X} = [x, y] \) is a point of the line then we have the parametric equation:
\[
\overrightarrow{X} = A + t\overrightarrow{AB},
\]
\[
[x, y] = [2, 3] + t[6, -8] = [2 + 6t, 3 - 8t].
\]
The slope of the line is \( -\frac{8}{6} = -\frac{4}{3} \), so the slope of the perpendicular bisector is \( \frac{3}{4} \)
and we may take as direction vector along the perpendicular bisector the vector \( \overrightarrow{V} = [4, 3] \).
The midpoint \( C \) of the segment \( AB \) is the point:
\[
C = \frac{1}{2}(A + B) = \frac{1}{2}([2, 3] + [8, -5]) = \frac{1}{2}([10, -2]) = [5, -1].
\]
So the required perpendicular bisector has the parametric equation:
\[
\overrightarrow{X} = C + s\overrightarrow{V},
\]
\[
[x, y] = [5, -1] + s[4, 3] = [5 + 4s, -1 + 3s].
\]
Let \([a, b]\) be a unit vector that makes 45 degrees with \(AB\). Then we have: 
\[\frac{6a - 8b}{\sqrt{6^2 + (-8)^2}} = \frac{6a - 8b}{\sqrt{36 + 64}} = \frac{6a - 8b}{10} = \frac{3a - 4b}{5},\]

\[3a - 4b = 5 \cos \left( \frac{\pi}{4} \right) = \frac{5}{\sqrt{2}},\]

\[a = \frac{1}{3} \left( 4b + \frac{5}{\sqrt{2}} \right),\]

\[1 = a^2 + b^2 = b^2 + \frac{1}{9} \left( 4b + \frac{5}{\sqrt{2}} \right)^2\]

\[9 = 9b^2 + 16b^2 + \frac{40}{\sqrt{2}} b + \frac{25}{2},\]

\[25b^2 + 20\sqrt{2} b = - \frac{7}{2},\]

\[(5b + 2\sqrt{2})^2 = 8 - \frac{7}{2} = \frac{9}{2},\]

\[5b = \sqrt{2} \left( -2 \pm \frac{3}{2} \right), \quad b = \frac{\sqrt{2}}{10} (-4 \pm 3),\]

\[a = \frac{1}{3} \left( 4 \left( \frac{\sqrt{2}}{10} (-4 \pm 3) \right) + \frac{5}{\sqrt{2}} \right)\]

\[= \frac{1}{15\sqrt{2}} (4(-4 \pm 3) + 25)\]

\[= \frac{1}{15\sqrt{2}} (9 \pm 12) = \frac{\sqrt{2}}{10} (3 \pm 4).\]

So the two unit vectors are:
\[
\frac{\sqrt{2}}{10} [3 + 4, -4 + 3], \quad \frac{\sqrt{2}}{10} [3 - 4, -4 - 3],
\]

\[
\frac{\sqrt{2}}{10} [7, -1] = [0.9899495, -0.1414214], \quad -\frac{\sqrt{2}}{10} [1, 7] = [-0.1414214, 0.9899495].
\]

Note that these are indeed unit vectors and they are mutually perpendicular, as they should be.
Alternatively and more easily, we first rescale the vector $\overrightarrow{AB}$ to a unit vector $\overrightarrow{N}$:

$$
\overrightarrow{N} = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{[6, -8]}{\sqrt{6^2 + (-8)^2}} = \frac{[6, -8]}{\sqrt{36 + 64}} = \frac{[6, -8]}{\sqrt{100}} = [0.6, -0.8].
$$

Then we rotate it counterclockwise through 45 degrees. The required rotation matrix is:

$$
\begin{bmatrix}
\cos \left( \frac{\pi}{4} \right) & -\sin \left( \frac{\pi}{4} \right) \\
\sin \left( \frac{\pi}{4} \right) & \cos \left( \frac{\pi}{4} \right)
\end{bmatrix}
= \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}
$$

Then the rotated vector is:

$$
\frac{1}{\sqrt{2}} \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix} \begin{bmatrix}
0.6 \\
-0.8
\end{bmatrix}
= \frac{1}{\sqrt{2}} \begin{bmatrix}
1.4 \\
-0.2
\end{bmatrix}
= \frac{\sqrt{2}}{10} \begin{bmatrix}
7 \\
-1
\end{bmatrix}
$$

So one of the required vectors is: $\frac{\sqrt{2}}{10} [7, -1]$, as before.
The other rotated vector is then obtained by using the inverse matrix, representing a forty-five degree clockwise rotation.

The inverse matrix is:

$$
\begin{bmatrix}
\cos \left( \frac{\pi}{4} \right) & \sin \left( \frac{\pi}{4} \right) \\
-\sin \left( \frac{\pi}{4} \right) & \cos \left( \frac{\pi}{4} \right)
\end{bmatrix}
= \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix}
$$

Then the rotated vector is:

$$
\frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix} \begin{bmatrix}
0.6 \\
-0.8
\end{bmatrix}
= \frac{1}{\sqrt{2}} \begin{bmatrix}
-0.2 \\
-1.4
\end{bmatrix}
= \frac{\sqrt{2}}{10} \begin{bmatrix}
1 \\
7
\end{bmatrix}
$$

So the other required vector is $-\frac{\sqrt{2}}{10} [1, 7]$, as before.
Question 5

The period $T$ of a pendulum of length $L$ is given by the formula:

$$T = 2\pi \sqrt{\frac{L}{g}}.$$

Here $g$ is the acceleration due to gravity.

Use the linear approximation to relate the fractional change in the period to the fractional change in the length.

In particular if the length is known an accuracy of within 0.1 percent, about how much possible error is there in the measurement of one day by the clock?

The differential $dT$ is given by:

$$dT = \frac{dT}{dL} dL = 2\pi \frac{1}{2\sqrt{Lg}} dL = \frac{\pi}{\sqrt{Lg}} dL.$$

Then we have:

$$\frac{dT}{T} = \frac{\frac{\pi}{\sqrt{Lg}} dL}{2\pi \sqrt{\frac{L}{g}}} = \frac{dL}{2L}.$$

We can also see this by taking logarithms a priori:

$$\ln(T) = \ln\left(2\pi \sqrt{\frac{L}{g}}\right) = \ln(2) + \ln(\pi) - \frac{1}{2} \ln(g) + \frac{1}{2} \ln(L).$$

Taking differentials, using $d \ln(u) = \frac{du}{u}$ immediately gives $\frac{dT}{T} = \frac{dL}{2L}$ (assuming that $g$ is constant, so $dg = 0$), as before.

Then the linear approximation gives, if $\Delta T$ is the error in $T$ and $\Delta L$ the error in $L$:

$$\frac{\Delta T}{T} = \frac{\Delta L}{2L}.$$

This equation says that the fractional change in $T$ is one-half the fractional change in $L$, for small changes.
So if the length is known to within 0.1 percent, then $T$ is known to within 0.05 percent.

A day has $24(60)(60) = 86400$ seconds, so 0.05 percent of a day is $\frac{86400}{20} = 43.2$ seconds.

So each day the clock could run fast or slow about 43.2 seconds.
Question 6

A cylindrical can of volume 1.2 liters is made from material costing 40 cents per square meter for the sides and base of the can and 100 cents per square meter for the top.

What are the dimensions of the can that minimize the total cost of materials for the can and what is that total cost?

Let the cylinder have radius \( x \) centimeters and height \( y \) centimeters.

Note that there are \( 100^2 \) square centimeters in a square meter.

- The volume \( V \) cubic centimeters of the cylinder is given by the formula \( V = \pi x^2 y \) (base area \( \pi x^2 \) times height \( y \)).
  
  Here \( V = 1200 \), so we have the equation:
  
  \[
  1200 = \pi x^2 y.
  \]

- The sides of the cylinder have area \( 2\pi xy \) (base circumference \( 2\pi x \) times height \( y \)).
  
  The base has area \( \pi x^2 \).
  
  So the total cost in cents of the base and sides is:
  
  \[
  C_1 = (2\pi xy + \pi x^2) \frac{40}{100^2} = \frac{\pi}{250} (x^2 + 2xy).
  \]

- The top of the cylinder has area \( \pi x^2 \).
  
  The cost of the top is:
  
  \[
  C_2 = \pi x^2 \frac{100}{100^2} = \frac{\pi}{100} (x^2).
  \]

So the total cost \( C \) cents of the materials is:

\[
C = C_1 + C_2 = \frac{\pi}{250} (x^2 + 2xy) + \frac{\pi}{100} (x^2) = \frac{\pi}{500} (2x^2 + 4xy + 5x^2)
\]

\[
= \frac{\pi}{500} (7x^2 + 4xy).
\]

But we have also \( 1200 = \pi x^2 y \), so \( y = \frac{1200}{\pi x^2} \).

So we may eliminate \( y \) from the formula for \( C \), giving:

\[
C = \frac{\pi}{500} \left(7x^2 + 4x \left( \frac{1200}{\pi x^2} \right)\right) = \frac{1}{500} \left(7\pi x^2 + 4800x^{-1}\right).
\]
As $x \to 0^+$ and as $x \to \infty$, $C \to \infty$.

So there exists at least one minimum in the interval $x > 0$.

At the minimum, the slope is zero, which gives the equation:

$$0 = C''(x) = \frac{1}{500}(14\pi x - 4800x^{-2})$$

$$14\pi x = 4800x^{-2},$$

$$x^3 = \frac{4800}{14\pi} = \frac{2400}{7\pi},$$

$$x = \sqrt[3]{\frac{2400}{7\pi}} = 2\sqrt[3]{\frac{300}{7\pi}} = 4.7788,$$

$$y = \frac{1200}{\pi x^2},$$

$$\frac{y}{x} = \frac{1200}{\pi x^3} = \frac{1200}{(\frac{2400}{7\pi})} = \frac{7}{2},$$

$$y = \frac{7}{2}x = 7\sqrt[3]{\frac{300}{7\pi}} = 16.7259.$$

Since this is the only possible minimum, it must give the required minimum cost.

So the minimum cost in cents is:

$$C = \frac{\pi}{500}(7x^2 + 4xy) = \frac{\pi}{500} \left( 7x^2 + 4x \left( \frac{7}{2}x \right) \right)$$

$$= \frac{21\pi x^2}{500}$$

$$= \frac{21\pi}{125} \left( \sqrt[3]{\frac{300}{7\pi}} \right)^2 = 3.0133.$$

So the radius for minimum cost is 4.7788 centimeters, the height is 16.7259 centimeters and the minimum cost is then 3.0133 cents.