Honors Calculus Homework 4 Solutions, due 9/29/5

Question 1

Calculate the derivative of each of the following functions:

• \( a(t) = \ln(\sin(t)) \).

We use the rules \( \ln'(t) = \frac{1}{t} \), \( \sin'(t) = \cos(t) \) and the chain rule formula:

\[
\frac{d}{dt} \ln(u(t)) = \frac{u'(t)}{u(t)};
\]

\[
a'(t) = \frac{1}{\sin(t)} \sin'(t) = \frac{\cos(t)}{\sin(t)} = \cot(t).
\]

• \( b(x) = \arcsin(x^2) \).

We use the rules \( \arcsin'(x) = \frac{1}{\sqrt{1-x^2}} \) and the chain rule:

\[
b'(x) = \frac{1}{\sqrt{1-(x^2)^2}} (2x) = \frac{2x}{\sqrt{1-x^4}}.
\]

• \( c(t) = \arctan(\sqrt{2t}) \).

We use the rules \( \arctan'(t) = \frac{1}{1+t^2} \) and the chain rule:

\[
c'(t) = \frac{1}{1+(\sqrt{2t})^2} \frac{d}{dt} (\sqrt{2t}) = \frac{(2t)^{-\frac{1}{2}}}{1 + 2t}.
\]

• \( p(x) = \ln(x + \sqrt{1+x^2}) \).

We have:

\[
p'(x) = \left( \frac{1}{x + \sqrt{1+x^2}} \right) \frac{d}{dx} (x + \sqrt{1+x^2})
\]

\[
= \left( \frac{1}{x + \sqrt{1+x^2}} \right) \left( 1 + \frac{x}{\sqrt{1+x^2}} \right)
\]

\[
= \left( \frac{1}{x + \sqrt{1+x^2}} \right) \left( \frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}} \right)
\]

\[
= \frac{1}{\sqrt{1+x^2}}.
\]
**Question 2**

Define a number $c$ by the formula:

$$c = \lim_{h \to 0} \frac{3^h - 1}{h}.$$

- Show that $c$ can be interpreted as the slope of the curve $y = 3^x$ at $x = 0$. If $f(x) = 3^x$, note that $f(0) = 3^0 = 1$.

Then its slope at the origin is, by definition:

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \frac{3^h - 3^0}{h} = \frac{3^h - 1}{h} = c.$$

- Show that:

$$2c = \lim_{h \to 0} \frac{9^h - 1}{h},$$

We have:

$$\lim_{h \to 0} \frac{9^h - 1}{h} = \lim_{h \to 0} \frac{3^{2h} - 1}{h} = \lim_{h \to 0} \frac{(3^h)^2 - 1}{h} = \lim_{h \to 0} \frac{(3^h - 1)(3^h + 1)}{h} = c(3^0 + 1) = 2c.$$

$$3c = \lim_{h \to 0} \frac{(27)^h - 1}{h},$$

We have:

$$\lim_{h \to 0} \frac{27^h - 1}{h} = \lim_{h \to 0} \frac{3^{3h} - 1}{h} = \lim_{h \to 0} \frac{(3^h)^3 - 1}{h} = \lim_{h \to 0} \frac{(3^h - 1)((3^h)^2 + 3^h + 1)}{h} = c((3^0)^2 + 3^0 + 1) = c(3^2 + 3^0 + 1) = 3c.$$

- Show that the function $f(x) = 3^x$ obeys the relation:

$$f'(x) = cf(x).$$

We have:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{3^{x+h} - 3^x}{h}$$

$$= \lim_{h \to 0} \frac{3^x(3^h - 1)}{h} = c(3^x) = cf(x).$$

- By using appropriate numerical estimation, or by plotting the graph $y = 3^x$ near $x = 0$, estimate the quantity $c$.

Using Maple, we estimate the slope to be approximately 1.099.
Question 3

Write out the proof that \( \lim_{x \to 5} x^2 = 25 \).

We first solve the inequality \(|x^2 - 25| < \epsilon\), given \(\epsilon > 0\), where without loss of generality, we take \(x > 0\) (since we need \(x\) near 5) and \(\epsilon < 25\):

\[
|x^2 - 25| < \epsilon,
-\epsilon < x^2 - 25 < \epsilon,
25 - \epsilon < x^2 < 25 + \epsilon,
\sqrt{25 - \epsilon} < x < \sqrt{25 + \epsilon}.
\]

For \(0 < \epsilon < 25\), put \(J_\epsilon = (\sqrt{25 - \epsilon}, \sqrt{25 + \epsilon})\).

Note that \(\sqrt{25 - \epsilon} < \sqrt{25} = 5 < \sqrt{25 + \epsilon}\), so 5 \(\in J_\epsilon\).

Then we have:

- \(J_\epsilon\) is an open interval.
- 5 \(\in J_\epsilon\).
- If \(x \in J_\epsilon\), then \(|x^2 - 25| < \epsilon\).

So by definition of the limit, we have \(\lim_{x \to 5} x^2 = 25\), as required.
Question 4

Let \( f(x) = \frac{x^3 - x^2 - x + 1}{x + 2} \).

- Give the domain of \( f(x) \).
  Explain why the range of \( f(x) \) is all real numbers.

The domain is \( \mathbb{R} - \{-2\} \): all real \( x \) not equal to \(-2\) or the union of the open intervals: \(( -\infty, -2) \cup (-2, \infty)\).

We have \((-2)^3 - (-2)^2 - (-2) + 1 = -8 - 4 + 2 + 1 = -9 < 0\), so as \( x \to -2^+ \), we have \( f(x) \to -\infty \) and as \( x \to -2^- \), we have \( f(x) \to \infty \). Also as \( x \to \infty \), we have \( f(x) \to \infty \).
Since \( f \) is continuous in the interval \((-2, \infty)\), it takes all real values on that interval, so its range is the whole real line.

- Find the derivative of \( f \) and find all points where the graph \( y = f(x) \) has a horizontal tangent.

We have:

\[
 f(x) = \frac{x^3 - x^2 - x + 1}{x + 2} = \frac{x^2(x - 1) - 1(x - 1)}{x + 2} = \frac{(x^2 - 1)(x - 1)}{x + 2}
\]

\[
 = (x + 1)(x - 1)^2(x + 2)^{-1},
\]

\[
f'(x) = (x-1)^2(x+2)^{-1}+2(x+1)(x-1)(x+2)^{-1}-(x+1)(x-1)^2(x+2)^{-2}
\]

\[
 = (x - 1)(x + 2)^{-1}(x - 1 + 2(x + 1) - \frac{(x + 1)(x - 1)}{x + 2})
\]

\[
 = (x - 1)(x + 2)^{-2}((3x + 1)(x + 2) - (x^2 - 1))
\]

\[
 = (x - 1)(x + 2)^{-2}(2x^2 + 7x + 3)
\]

\[
 = \frac{(x - 1)(2x + 1)(x + 3)}{(x + 2)^2}.
\]

So the horizontal tangents occur at \((1, 0)\), \((-3, 32)\) and at \((-\frac{1}{2}, \frac{3}{4})\).
• Plot the graph of the function $y = f(x)$.

The graph decreases steadily until it reaches its local minimum at $(-3, 32)$, then increases to infinity $x \to -2^-$.
It is concave up for $x < -2$. For $x > -2$, it increases from $-\infty$ as $x \to -2^+$, until it reaches a local maximum at $(-\frac{1}{2}, \frac{3}{4})$.
Then it decreases to its local minimum at $(1, 0)$.
Then it increases steadily, going to infinity as $x \to \infty$.
Using Maple we find that the second derivative vanishes only for $x = -2 + \sqrt{9}$, when $y = 15 - 7\sqrt{9}$, so at the point $(0.080083823, 0.43941324)$.
In the intervals $x < -2$ and $x > -2 + \sqrt{9}$, the graph is concave up.
In the interval $(-2, -2 + \sqrt{9})$, the graph is concave down.

• Find the largest domain for $f$ of the form $(a, \infty)$ for a suitable real number $a$, such that $f$ has an inverse on that domain, explaining your answer.

The number $a$ is the $x$-value for the local minimum, namely $a = 1$, since $f(x)$ is increasing strictly on the domain $[1, \infty)$, but because $a = 1$ is a local minimum, $f(x)$ is not one to one on any interval of the form $(b, \infty)$ for $b < 1$. 
Question 5

Let \( g(t) = \frac{1}{8}(t^3 - 12t) \).

- Plot the graph of the function \( g \).

We have:

\[
- g = \frac{1}{8}(t^3 - 12t) = \frac{1}{8}(t)(t^2 - 12) = \frac{1}{8}(t)(t - 2\sqrt{3})(t + 2\sqrt{3}),
\]

\[
- g' = \frac{3}{8}(t^2 - 4) = \frac{3}{8}(t - 2)(t + 2),
\]

\[
- g'' = \frac{3}{4} t.
\]

This is a standard cubic curve, shaped like the letter s.

It increases, passing through the \( t \) axis at \( t = -\sqrt{12} = -2\sqrt{3} \) until \( t = -2 \), where it reaches a local maximum of 2, then decreases, crossing the axes at the origin, until it reaches a local minimum of \( -2 \) at \( t = 2 \).

Then it increases again crossing the \( t \)-axis again at \( t = \sqrt{12} = 2\sqrt{3} \). As \( t \to \pm \infty \), we have \( g(t) \to \pm \infty \).

The graph is concave down for \( t \leq 0 \) and concave up for \( t \geq 0 \).

- Explain why the function \( g \) has an inverse if its domain is restricted to the interval \( J = (-2, 2) \).

What is the range \( K \) of \( g \) on the domain \( J \)?

Explain.

On the interval \( J \), we have \( g'(t) < 0 \), so the function is strictly decreasing and so has an inverse.

Since \( g \) is continuous and decreasing on \( J \), the range is:

\[
K = (g(2), g(-2)) = (-2, 2).
\]

- Let \( g^{-1} : K \to J \) be the inverse function.
• Find the equation of the tangent line to the function \( y = g(t) \) at the point with \( t = 1 \).

We have:

\[
g(1) = \frac{1}{8}(1^3 - 12(1)) = -\frac{11}{8},
\]

\[
g'(1) = \frac{3}{8}(1^2 - 4(1)) = -\frac{9}{8}.
\]

So by the point-slope method the required tangent line is:

\[
y - (-\frac{11}{8}) = -\frac{9}{8}(t - 1),
\]

\[
8y + 11 = -9t + 9,
\]

\[
8y + 9t + 2 = 0.
\]

• Find the equation of the tangent line to the function \( y = g^{-1}(t) \) at the point with \( t = -\frac{11}{8} \).

Since \( g(1) = \frac{11}{8} \), the required tangent line is simply the reflection of the tangent line to the curve \( y = g(t) \) at \( t = 1 \) in the axis \( y = t \).

So the required equation is obtained by interchanging \( y \) and \( t \) in the equation \( 8y + 9t + 2 = 0 \), so is: \( 8t + 9y + 2 = 0 \).

• Sketch the functions \( y = g(t) \) and \( y = g^{-1}(t) \) and the two tangent lines on one graph, using the same scaling for each axis and discuss your results.
Question 6

Let \( A = [2, 5], \ B = [-3, -7] \) and \( C = [14, 0] \).

- Sketch the triangle \( ABC \).
- Find the lengths of the sides of the triangle \( ABC \).

We have:
\[
BC = C - B = [14, 0] - [-3, -7] = [17, 7],
\]
\[
CA = A - C = [2, 5] - [14, 0] = [-12, 5],
\]
\[
\]

Then the lengths of the sides are:

\[
a = |BC| = ||17, 7|| = \sqrt{17^2 + 7^2} = \sqrt{289 + 49} = \sqrt{338} = \sqrt{(169)(2)} = 13\sqrt{2}
\]

\[
b = |CA| = ||-12, 5|| = \sqrt{(-12)^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13,
\]

\[
c = |CA| = ||-5, -12|| = \sqrt{(-5)^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13.
\]

- Find the angles at the vertices of the triangle \( ABC \).

We notice that \( b = c \), so the triangle is isosceles.
Also \( a^2 = 338 = 169 + 169 = b^2 + c^2 \), so the triangle obeys Pythagoras’ so is right-angled, with hypotenuse \( a \).
The only right-angled isosceles triangle has one angle of 90 degrees and two of 45 degrees.
So the angle at \( A \) is 90 degrees and the angles at \( B \) and \( C \) are each forty-five degrees.
Note that the slope of \( CA \) is \( m_b = -\frac{5}{12} \) and of \( AB \) is \( m_c = \frac{12}{5} \).
Since \( m_b m_c = -1 \), these lines are perpendicular, in agreement with the fact that the angle at \( A \) is 90 degrees.

- Find the area of the triangle \( ABC \).

Since the triangle is right-angled at \( A \), its area \( \Delta \) in units of area is:
\[
\Delta = \frac{1}{2} bc \sin(A) = \frac{1}{2} bc \sin\left(\frac{\pi}{2}\right) = \frac{1}{2} bc = \frac{1}{2} 13^2 = \frac{169}{2} = 84.5.
\]