Honors Calculus Homework 11 Solutions, due 11/17/5

Question 1

Find the Taylor series of $f(x) = \sin(3x)$, based at the origin.
Use it to estimate $f(0.1) = \sin(0.3)$, including enough terms to make sure that the error in your approximation is less than $10^{-10}$.

We have:

$$f(x) = \sin(3x), f'(x) = 3\cos(3x), f''(x) = -9\sin(3x),$$
$$f'''(x) = -27\cos(3x), f''''(3x) = 81\sin(3x) = 3^4 f(x).$$

By induction, we have:

$$f^{(4n)}(x) = 3^{4n} f(x) = 3^{4n} \sin(3x),$$
$$f^{(4n+1)}(x) = 3^{4n} f'(x) = 3^{4n+1} \cos(3x),$$
$$f^{(4n+2)}(x) = 3^{4n} f''(x) = -3^{4n+2} \sin(3x),$$
$$f^{(4n+3)}(x) = 3^{4n} f'''(x) = -3^{4n+3} \cos(3x).$$

Evaluating at the base point $x = 0$, we get:

$$f^{(4n)}(0) = 0,$$
$$f^{(4n+1)}(0) = 3^{4n+1},$$
$$f^{(4n+2)}(0) = 0,$$
$$f^{(4n+3)}(0) = -3^{4n+3}.$$

So we have $f^{(2n)}(0) = 0$ and $f^{(2n+1)}(0) = 3^{2n+1} (-1)^n$.

Then the Taylor series of $f(x)$ based at the origin is:

$$T_{\infty}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)x^{2n+1}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!}.$$
When $x = 0.1$, the fifth term in this series (so the term with $n = 4$), has size:

$$\frac{3^9 \cdot 1}{10^9 \cdot 9!} = \frac{3^9}{9(8)(7)(6)(5)(4)(3)(2)(10^9)}$$

$$= \frac{243}{(448)10^{10}} < 10^{-10}.$$

The terms in the series alternate in size, so the error in taking the first four terms is less than the fifth term, which gives us the desired accuracy.

So the approximation with the correct accuracy is:

$$\left[ 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} \right]_{x=0.1}$$

$$= \left[ 3x \left( 1 - \frac{3x^2}{2} + \frac{27x^4}{40} - \frac{81x^6}{560} \right) \right]_{x=0.1}$$

$$= \frac{3}{10} \left( 1 - \frac{3}{200} + \frac{27}{400000} - \frac{81x^6}{560000000} \right)$$

$$= \frac{3}{56(10^8)} (56(10^7) - 3(28)(10^5) + 27(1400) - 81)$$

$$= \frac{3}{56(10^8)} (560037800 - 8400081)$$

$$= \frac{3}{56(10^8)} (551637719).$$

According to Maple, the factor 551637719 is prime, so we cannot simplify further.

If we evaluate this as a decimal, we get the estimate:

$$0.29552020660714285.$$

The value of $\sin(0.3)$ according to Maple is:

$$0.2955202066613395751053207457.$$

The error, $5.4196718(10^{-11})$, is within the desired limit of $10^{-10}$, as required.
Question 2

Estimate the integral \( \int_0^{\pi} \sin(x) dx \), using 6 intervals and the following rules:

- The trapezoidal rule.
- The midpoint rule.
- Simpson’s rule.

With \( f(x) = \sin(x) \), the data for the trapezoidal and Simpson’s rules are:

\[ y_0 = f(0) = \sin(0) = 0, \]
\[ y_1 = f \left( \frac{\pi}{12} \right) = \sin \left( \frac{\pi}{12} \right) = \frac{1}{2} \sqrt{2 - \sqrt{3}}, \]
\[ y_2 = f \left( \frac{\pi}{6} \right) = \sin \left( \frac{\pi}{6} \right) = \frac{1}{2}, \]
\[ y_3 = f \left( \frac{\pi}{4} \right) = \sin \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2}, \]
\[ y_4 = f \left( \frac{\pi}{3} \right) = \sin \left( \frac{\pi}{3} \right) = \frac{\sqrt{3}}{2}, \]
\[ y_5 = f \left( \frac{5\pi}{12} \right) = \sin \left( \frac{5\pi}{12} \right) = \frac{1}{2} \sqrt{2 + \sqrt{3}}, \]
\[ y_6 = f \left( \frac{\pi}{2} \right) = \sin \left( \frac{\pi}{2} \right) = 1. \]

We have \( \Delta x = \frac{\pi}{12} \), so the trapezoidal rule estimate is:

\[ T_6 = \frac{\Delta x}{2} (y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + y_6) \]
\[ = \frac{\pi}{24} \left( 2 + \sqrt{2} + \sqrt{3} + \sqrt{2 - \sqrt{3}} + \sqrt{2 + \sqrt{3}} \right) \]
\[ = 0.994281888. \]
Simpson’s rule estimate is:

\[ S_6 = \frac{\Delta x}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6) \]

\[
= \frac{\pi}{36} \left( 2 + 2\sqrt{2} + \sqrt{3} + 2\sqrt{2 - \sqrt{3}} + 2\sqrt{2 + \sqrt{3}} \right) = 1.000026312.
\]

The exact result is by FTCII:

\[
\int_0^{\pi} \sin(x) \, dx = -[\cos(x)]_0^\pi = -\cos \left( \frac{\pi}{2} \right) - (-\cos(0)) = 1.
\]

Clearly Simpson’s rule gives a more accurate estimate in this case.

The data for the mid-point rule are:

- \( m_1 = f \left( \frac{\pi}{24} \right) = \sin \left( \frac{\pi}{24} \right) \),
- \( m_2 = f \left( \frac{3\pi}{24} \right) = \sin \left( \frac{\pi}{8} \right) \),
- \( m_3 = f \left( \frac{5\pi}{24} \right) = \sin \left( \frac{5\pi}{24} \right) \),
- \( m_4 = f \left( \frac{7\pi}{24} \right) = \sin \left( \frac{7\pi}{24} \right) \),
- \( m_5 = f \left( \frac{9\pi}{24} \right) = \sin \left( \frac{3\pi}{8} \right) \),
- \( m_6 = f \left( \frac{11\pi}{24} \right) = \sin \left( \frac{11\pi}{24} \right) \).

Using trigonometric identities it can be shown (see below) that the sum:

\[ M = m_1 + m_2 + m_3 + m_4 + m_5 + m_6 \]

simplifies to just:

\[ M = \frac{1}{2 \sin \left( \frac{\pi}{24} \right)}. \]

Then the midpoint rule sum is:

\[ M_6 = \Delta x(m_1 + m_2 + m_3 + m_4 + m_5 + m_6) = M \Delta x \]

\[
= \frac{\pi}{12} \left( \frac{1}{2 \sin \left( \frac{\pi}{24} \right)} \right) = \frac{\pi}{24 \sin \left( \frac{\pi}{24} \right)} = 1.0028615.
\]
To simplify the sum \( M = m_1 + m_2 + \cdots + m_6 \) we use Euler’ s formulas:

\[
e^{it} = \cos(t) + i \sin(t),
\]

\[
e^{a+b} = e^a e^b.
\]

\[
(e^a)^n = e^{an},
\]

\[
e^{i \frac{\pi}{2}} = \cos \left( \frac{\pi}{2} \right) + i \sin \left( \frac{\pi}{2} \right) = 0 + i(1) = i.
\]

Then we want for the sum \( M \) the imaginary part of the following sum, which we sum using the finite geometric series formula (where the ratio \( r \) is \( e^{i \frac{\pi}{24}} \)):

\[
e^{i \frac{\pi}{24}} + e^{i \frac{3\pi}{24}} + e^{i \frac{5\pi}{24}} + e^{i \frac{7\pi}{24}} + e^{i \frac{9\pi}{24}} + e^{i \frac{11\pi}{24}}
\]

\[
= e^{i \frac{\pi}{24}} \left( 1 + e^{i \frac{\pi}{12}} + (e^{i \frac{\pi}{12}})^2 + (e^{i \frac{\pi}{12}})^3 + (e^{i \frac{\pi}{12}})^4 + (e^{i \frac{\pi}{12}})^5 \right)
\]

\[
= e^{i \frac{\pi}{24}} \left( \frac{1 - (e^{i \frac{\pi}{12}})^6}{1 - e^{i \frac{\pi}{12}}} \right)
\]

\[
= \left( \frac{1 - e^{i \frac{\pi}{2}}}{e^{-i \frac{\pi}{24}} - e^{i \frac{\pi}{24}}} \right)
\]

\[
= \left( \frac{1 - i}{-2i \sin \left( \frac{\pi}{24} \right)} \right)
\]

\[
= \frac{1 + i}{2 \sin \left( \frac{\pi}{24} \right)}.
\]

The imaginary part of this expression gives immediately the required relation:

\[
M = \frac{1}{2 \sin \left( \frac{\pi}{24} \right)}.
\]
For each use the error estimation formula to estimate the error and for each compare with the exact result of the integration.

- The trapezoidal rule error formula for an integral $\int_{a}^{b} f(x)\,dx$ computed using the trapezoidal rule with $n$ equal intervals is:

$$E_T = \frac{K_2 (b - a)^3}{12n^2}.$$ 

Here $K_2$ is the maximum of $|f''(x)|$ on the interval $[a, b]$. Here $b = \frac{\pi}{2}$, $a = 0$, $n = 6$. Also $f(x) = \sin(x)$, so $|f''(x)| = |\sin(x)|$ which achieves its maximum value of 1 at $x = \frac{\pi}{2}$. So $K_2 = 1$. So we have:

$$E_T = \frac{K_2 (b - a)^3}{12n^2} = \frac{1 \left( \frac{\pi^3}{8} \right)}{12(36)} = \frac{\pi^3}{3456} = 0.0089926.$$ 

The actual error is $0.0057181$, well within the maximum possible error.

- The midpoint rule error formula for an integral $\int_{a}^{b} f(x)\,dx$ computed using the midpoint rule with $n$ equal intervals is:

$$E_M = \frac{K_2 (b - a)^3}{24n^2}.$$ 

Here $K_2$ is the maximum of $|f''(x)|$ on the interval $[a, b]$. Here $b = \frac{\pi}{2}$, $a = 0$, $n = 6$. Also $f(x) = \sin(x)$, so $|f''(x)| = |\sin(x)|$ which achieves its maximum value of 1 at $x = \frac{\pi}{2}$. So $K_2 = 1$. So we have:

$$E_M = \frac{K_2 (b - a)^3}{24n^2} = \frac{1 \left( \frac{\pi^3}{8} \right)}{24(36)} = \frac{\pi^3}{6912} = 0.0044963.$$ 

The actual error is $0.0028615$, well within the maximum possible error.
The Simpson’s rule error formula for an integral \( \int_a^b f(x)dx \) computed using Simpson’s rule with an even number \( n \) equal intervals is:

\[
\mathcal{E}_S = \frac{\mathcal{K}_4 (b-a)^5}{180n^4}.
\]

Here \( \mathcal{K}_4 \) is the maximum of \( |f'''(x)| \) on the interval \([a, b]\).

Here \( b = \frac{\pi}{2}, a = 0, n = 6 \).

Also \( f(x) = \sin(x) \), so \( |f'''(x)| = |\sin(x)| \) which achieves its maximum value of 1 at \( x = \frac{\pi}{2} \).

So \( \mathcal{K}_4 = 1 \).

So we have:

\[
\mathcal{E}_S = \frac{\mathcal{K}_4 (b-a)^5}{180n^4} = \frac{1 \left( \frac{\pi^5}{32}\right)}{180(1296)} = \frac{\pi^5}{7464960} = 0.000040995.
\]

The actual error is 0.000026312, well within the maximum possible error.
Question 3

Let \( f(x) = x^2 - 5x + 6 \) and \( g(x) = x^3 - x^2 - 10x + 12 \).

Carefully sketch the region enclosed by the curves \( y = f(x) \) and \( y = g(x) \) and find the total area of the region.

The two given curves meet where \( f(x) = g(x) \), so where:

\[
x^2 - 5x + 6 = x^3 - x^2 - 10x + 12,
\]
\[
0 = x^3 - 2x^2 - 5x + 6,
\]
\[
0 = (x - 1)(x^2 - x - 6) = (x - 1)(x - 3)(x + 2).
\]

When \( x = 1 \) the curves meet at \( (1, 2) \), when \( x = 3 \), the curves meet at \( (3, 0) \) and when \( x = -2 \) the curves meet at \( (-2, 20) \).

Plotting shows that the cubic lies above the parabola in the interval \((-2, 1)\) and below in the interval \((1, 3)\).

So the required area is:

\[
\int_{-2}^{3} |f(x) - g(x)|\,dx = \int_{-2}^{3} |x^3 - 2x^2 - 5x + 6|\,dx
\]
\[
= \int_{-2}^{1} (x^3 - 2x^2 - 5x + 6)\,dx + \int_{1}^{3} -(x^3 - 2x^2 - 5x + 6)\,dx
\]
\[
= \left[ \frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x \right]_{-2}^{1} - \left[ \frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x \right]_{1}^{3}
\]
\[
= - \left( \frac{16}{4} + \frac{16}{3} - \frac{20}{2} - 12 \right) + 2 \left( \frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6 \right) - \left( \frac{81}{4} - \frac{54}{3} - \frac{45}{2} + 18 \right)
\]
\[
= \frac{38}{3} + \frac{37}{6} + \frac{9}{4}
\]
\[
= \frac{1}{12}(152 + 74 + 27)
\]
\[
= \frac{253}{12} = 21.083.
\]
Question 4

Let \( F(x) = \int_{x^2}^{\sin(x)} 4te^{-t^2} \, dt \).

Find \( F'(x) \) in two ways:

- By first doing the integral exactly and then differentiating.

  We have by FTCII, since \( \frac{d}{dt} e^{-t^2} = -2te^{-t^2} \):

  \[
  \int 4te^{-t^2} \, dt = -2e^{-t^2} + C,
  \]

  \[
  F(x) = \int_{x^2}^{\sin(x)} 4te^{-t^2} \, dt = \left[ -2e^{-t^2} \right]_{x^2}^{\sin(x)} = -2e^{-\sin^2(x)} + 2e^{-x^4}.
  \]

  Differentiating, we get:

  \[
  F'(x) = 4 \sin(x) \cos(x)e^{-\sin^2(x)} - 8x^3e^{-x^4}.
  \]

- By using the Fundamental Theorem of Calculus and the chain rule.

  By FTCI and the chain rule, if \( F(x) = \int_{v(x)}^{u(x)} f(t) \, dt \) then we have:

  \[
  F'(x) = f(u(x))u'(x) - f(v(x))v'(x).
  \]

  Here we have:

  - \( u(x) = \sin(x) \), \( u'(x) = \cos(x) \),
  - \( v(x) = x^2 \), \( v'(x) = 2x \),
  - \( f(t) = 4te^{-t^2} \).

  So we get:

  \[
  F'(x) = (4te^{-t^2})|_{t=\sin(x)}(\cos(x)) - (4te^{-t^2})|_{t=x^2}(2x)
  = 4 \sin(x) \cos(x)e^{-\sin^2(x)} - 8x^3e^{-x^4}.
  \]

  Note that the two different approaches to calculating \( F'(x) \) agree, as they should.
Question 5

Write out an appropriate substitution formula for each of the following integrals and hence evaluate each integral:

\[ \int_{0}^{5} (2 - x)e^{x^2 - 4x + 5} \, dx. \]

- We substitute \( u = x^2 - 4x + 5 \).
- Then we have: \( du = \frac{du}{dx} \, dx = (2x - 4) \, dx = 2(x - 2) \, dx \)
- The \( u \)-range is: from \( u = 5 \) to \( u = 10 \).
- The differential \((2 - x)e^{x^2 - 4x + 5} \, dx\) becomes: \(-\frac{1}{2}e^u \, du\)

Hence the integral becomes:
\[
\int_{5}^{10} -\frac{1}{2}e^u \, du = -\frac{1}{2} [e^u]_{5}^{10} = -\frac{1}{2}(e^{10} - e^{5}) = -10939.02632.
\]

\[ \int_{0}^{5} \frac{x + 1}{x^2 - 4x + 5} \, dx. \]

Completing the square, we note that:
\[ x^2 - 4x + 5 = x^2 - 4x + 4 + 1 = (x - 2)^2 + 1. \]
- We substitute \( u = x - 2 \), so \( x = u + 2 \).
- Then we have: \( du = \frac{du}{dx} \, dx = dx \)
- The \( u \)-range is: from \( u = -2 \) to \( u = 3 \).
- The differential \( \frac{x + 1}{x^2 - 4x + 5} \, dx \) becomes: \( \frac{x + 1}{u^2 + 1} \, du \)

Hence the integral becomes:
\[
\int_{-2}^{3} \frac{u + 3}{u^2 + 1} \, du = \int_{-2}^{3} \left( \frac{u}{u^2 + 1} + \frac{3}{u^2 + 1} \right) \, du = \left[ \frac{1}{2} \ln(u^2 + 1) + 3 \text{arctan}(u) \right]_{-2}^{3}
\]
\[
= \frac{1}{2} \left( \ln(10) - \ln(5) \right) + 3(\text{arctan}(3) - \text{arctan}(-2))
\]
\[
= \frac{1}{2} \ln(2) + 3 \text{arctan}(3) + 3 \text{arctan}(2) = \frac{1}{2} + \frac{9}{4} \pi = 7.415157060857007.
\]
Question 6

By using an appropriate technique, determine the following integrals:

- \[ \int_0^2 \frac{1}{\sqrt{16 + 6x - x^2}} \, dx. \]

We note that by completing the square, we have:

\[ 16 + 6x - x^2 = 16 - (x^2 - 6x) = 25 - (x^2 - 6x + 9) = 25 - (x - 3)^2. \]

To be able to take the square root we want this expression to read:

\[ 25 - 25 \sin^2(t) = 25(1 - \sin^2(t)) = 25 \cos^2(t). \]

So we want \((x - 3)^2 = 25 \sin^2(t)\).

Therefore we make the substitution:

- \( x - 3 = 5 \sin(t), \quad x = 3 + 5 \sin(t), \quad t = \arcsin \left( \frac{x - 3}{5} \right) \).

- \( dx = \frac{dx}{dt} \, dt = 5 \cos(t) \, dt. \)

- \( 16 + 6x - x^2 = 25 - (x - 3)^2 = 25 - 25 \sin^2(t) = 25(1 - \sin^2(t)) = 25 \cos^2(t). \)

- \( \sqrt{16 + 6x - x^2} = 5 \cos(t). \)

- The \( t \) range is from \( t = -\arcsin \left( \frac{3}{5} \right) \) to \( t = -\arcsin \left( \frac{1}{5} \right) \).

- The integrand becomes:

\[ \frac{1}{\sqrt{16 + 6x - x^2}} \, dx = \frac{5 \cos(t) \, dt}{5 \cos(t)} = dt. \]

So the transformed integral is:

\[ \int_0^2 \frac{1}{\sqrt{16 + 6x - x^2}} \, dx = \int_{-\arcsin \left( \frac{3}{5} \right)}^{-\arcsin \left( \frac{1}{5} \right)} dt = \left[ \arcsin \left( \frac{x}{5} \right) \right]_{-\arcsin \left( \frac{3}{5} \right)}^{-\arcsin \left( \frac{1}{5} \right)} = \arcsin \left( \frac{3}{5} \right) - \arcsin \left( \frac{1}{5} \right) = \arcsin \left( \frac{1}{25} (6 \sqrt{7} (6) - 4)) \right) \]

\[ = 0.442143188003. \]
\[ \int_0^2 \frac{x^3 - 13x^2 + 11x - 5}{x^2 - 7x + 12} \, dx. \]

We first divide \( x^2 - 7x + 12 \) into \( x^3 - 13x^2 + 11x - 5 \).

The quotient is \( x - 6 \), with remainder \( r \) given as follows:

\[ r = x^3 - 13x^2 + 11x - 5 - (x - 6)(x^2 - 7x + 12) \]
\[ = x^3 - 13x^2 + 11x - 5 - (x^3 - 13x^2 + 54x - 72) = -43x + 67. \]

So we have:

\[ J = \int_0^2 \frac{x^3 - 13x^2 + 11x - 5}{x^2 - 7x + 12} \, dx = \int_0^2 (x - 6 + \frac{-43x + 67}{x^2 - 7x + 12}) \, dx \]
\[ = \left[ \frac{x^2}{2} - 6x \right]_0^2 + K = \left( \frac{2^2}{2} - 6(2) \right) - 0 + K = K - 10, \]

\[ K = \int_0^2 \frac{-43x + 67}{x^2 - 7x + 12} \, dx. \]

For the integral \( K \) we use partial fractions:

\[ \frac{-43x + 67}{x^2 - 7x + 12} = \frac{-43x + 67}{(x - 3)(x - 4)} = \frac{A}{x - 3} + \frac{B}{x - 4}. \]

\[ -43x + 67 = A(x - 4) + B(x - 3). \]

Putting \( x = 3 \) gives: \(-A = -43(3) + 67 = -129 + 67 = -62\), so \( A = 62 \).

Putting \( x = 4 \) gives: \( B = -43(4) + 67 = -172 + 67 = -105 \).

So the integral becomes:

\[ K = \int_0^2 \left( \frac{62}{x - 3} - \frac{105}{x - 4} \right) \, dx \]
\[ = [62 \ln(|x - 3|) - 105 \ln(|x - 4|)]_0^2 \]
\[ = 62(\ln(1) - \ln(3)) - 105(\ln(2) - \ln(4)) \]
\[ = -62 \ln(3) + 105 \ln(2). \]

So the required integral is:

\[ J = K - 10 = 105 \ln(2) - 62 \ln(3) - 10 = -5.3335079386285433777. \]
\[ \int_3^4 x^{1002}(4x - x^3)^{-1001} \, dx. \]

We first factor:

\[
\int_3^4 x^{1002}(4x - x^3)^{-1001} \, dx = \int_3^4 x^{1002}(4 - x^2)^{-1001} \, dx \\
= \int_3^4 (4 - x^2)^{-1001} \, dx.
\]

Now we substitute:

- \( u = 4 - x^2 \).
- \( du = \frac{du}{dx} \, dx = -2 \, dx \).
- The \( u \)-range is from \( u = -5 \) to \( u = -12 \).
- The integrand becomes \( x(4 - x^2)^{-1001} \, dx = -\frac{1}{2}u^{-1001} \, du \).

So the integral becomes:

\[
\int_{-5}^{-12} -\frac{1}{2}u^{-1001} \, du \\
= \frac{1}{2000} [u^{-1000}]_{-5}^{-12} \\
= -\frac{1}{2000} (5^{-1000} - 12^{-1000}) = -5.357543036(10^{-703}).
\]