Honors Calculus Notes 8/29/5

The derivative

If \( f(t) \) is a function of the real variable \( t \), its derivative \( f'(a) \) at a point \( t = a \) of its domain is given by the formulas:

\[
f'(a) = \lim_{t \to a} \frac{f(t) - f(a)}{t - a}
\]

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

If the limit in question does not exist at \( t = a \) then \( f \) is not differentiable at \( t = a \). If the limit in question does exist at \( t = a \) then \( f \) is differentiable at \( t = a \).

The derivative function \( f'(t) \) is given by the formula:

\[
f'(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t} = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h}.
\]

Examples

- \( f(t) = t^2 \):
  \[
f'(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t} = \lim_{s \to t} \frac{s^2 - t^2}{s - t} = \lim_{s \to t} (s - t)(s + t) = \lim_{s \to t} (s + t) = 2t.
\]

  Here \( f \) is everywhere differentiable.

- \( f(t) = t^2 \):
  \[
f'(t) = \lim_{h \to 0} \frac{f(h + t) - f(t)}{h} = \lim_{h \to 0} \frac{(h + t)^2 - t^2}{h}
  \]
  \[
  = \lim_{h \to 0} \frac{h^2 + t^2 + 2th - t^2}{h} = \lim_{h \to 0} \frac{h^2 + 2th}{h} = \lim_{h \to 0} h + 2t = 2t.
\]

  Here \( f \) is everywhere differentiable.
• \( f(t) = \frac{1}{t} \) (where \( t \) is any non-zero real number):

\[
f'(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t} = \lim_{s \to t} \frac{\frac{1}{s} - \frac{1}{t}}{s - t} = \lim_{s \to t} \frac{t - s}{st(s - t)}
\]

\[
= \lim_{s \to t} \frac{-1}{st} = \frac{-1}{t^2} = -t^{-2}.
\]

Here \( f \) is everywhere differentiable on its domain.

• \( f(t) = \frac{1}{t^2} \) (where \( t \) is any non-zero real number):

\[
f'(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t} = \lim_{s \to t} \frac{\frac{1}{s^2} - \frac{1}{t^2}}{s - t} = \lim_{s \to t} \frac{t^2 - s^2}{s^2 t^2 (s - t)}
\]

\[
= \lim_{s \to t} \frac{(t - s)(t + s)}{s^2 t^2 (s - t)} = \lim_{s \to t} \frac{-(t + s)}{s^2 t^2} = -\frac{2t}{t^4} = -\frac{2}{t^3}.
\]

Here \( f \) is everywhere differentiable on its domain.

• \( f(t) = t^{\frac{1}{2}} \) (where \( t \) is any non-negative real number):

If \( t > 0 \), we have:

\[
f'(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t} = \lim_{s \to t} \frac{s^{\frac{1}{2}} - t^{\frac{1}{2}}}{s - t} = \lim_{s \to t} \frac{s^{\frac{1}{2}} - t^{\frac{1}{2}}}{(s^{\frac{1}{2}} - t^{\frac{1}{2}})(s^{\frac{1}{2}} + t^{\frac{1}{2}})}
\]

\[
= \lim_{s \to t} \frac{1}{(s^{\frac{1}{2}} + t^{\frac{1}{2}})} = \frac{1}{2t^{\frac{1}{2}}}.
\]

Here the derivative \( f'(0) \) does not exist:

\[
f'(0) = \lim_{s \to 0^+} \frac{f(s) - f(0)}{s - 0} = \lim_{s \to 0^+} \frac{s^{\frac{1}{2}}}{s} = \lim_{s \to 0^+} \frac{1}{s^{\frac{1}{2}}} = \infty.
\]

So in this case the function is differentiable on a smaller domain than the domain of the function itself.
\( f(t) = |t| \).

- If \( t > 0 \), we have \( f(t) = t \) and \( f'(t) = \lim_{s\to t} \frac{s - t}{s - t} = 1 \).

- If \( t < 0 \), we have \( f(t) = -t \) and \( f'(t) = \lim_{s\to t} \frac{-s - (-t)}{s - t} = -1 \).

- Finally \( f'(0) = \lim_{s\to 0} \frac{|s|}{s} \).
  But we have:
  \[ \lim_{s\to 0^+} \frac{|s|}{s} = \lim_{s\to 0^+} \frac{s}{s} = 1, \quad \lim_{s\to 0^-} \frac{|s|}{s} = \lim_{s\to 0^-} \frac{-s}{s} = -1. \]
  A limit \( \lim_{s\to a} \) exists if both the limits \( \lim_{s\to a^+} \) and \( \lim_{s\to a^-} \) exist and are equal and then all three limits are equal.
  In the case of \( f'(0) \) the limits \( \lim_{s\to 0^+} \) and \( \lim_{s\to 0^-} \) exist, but are unequal, so the derivative \( f'(0) \) does not exist.

- \( f(t) = t^n \), where \( n \) is a positive integer.

We need the factorization:
\[ s^n - t^n = (s - t)(s^{n-1} + ts^{n-2} + t^2 s^{n-3} + \ldots + t^{n-2} s + t^{n-1}) \]
This can be proved by multiplying out the right-hand side.

- The term \( t^r s^{n-r-1} \) when multiplied by \( s \) gives \( t^r s^{n-r} \).
- The term \( t^{r-1} s^{n-r} \) when multiplied by \( -t \) gives \( -t^r s^{n-r} \).

So the terms cancel in pairs, except for the last term \( -t(t^{n-1}) = -t^n \) and the first term \( s(s^{n-1}) = s^n \).

So the two remaining terms combine to give \( s^n - t^n \) which is exactly the left-hand side.

So now we have:
\[
\begin{align*}
  f'(t) &= \lim_{s\to t} \frac{f(s) - f(t)}{s - t} = \lim_{s\to t} \frac{s^n - t^n}{s - t} \\
  &= \lim_{s\to t} \frac{(s - t)(s^{n-1} + ts^{n-2} + t^2 s^{n-3} + \ldots + t^{n-2} s + t^{n-1})}{(s - t)} \\
  &= \lim_{s\to t} (s^{n-1} + ts^{n-2} + t^2 s^{n-3} + \ldots + t^{n-2} s + t^{n-1}) \\
  &= t^{n-1} + t^{n-1} + t^{n-1} + \ldots + t^{n-1} + t^{n-1} = nt^{n-1}.
\end{align*}
\]
• \( g(t) = \frac{1}{f(t)} \) (considered on the domain where \( f(t) \neq 0 \)).

Here we assume that \( f \) is differentiable.

\[
g'(t) = \lim_{s \to t} \frac{g(s) - g(t)}{s - t} = \lim_{s \to t} \frac{\frac{1}{f(s)} - \frac{1}{f(t)}}{s - t}
\]

\[
= \lim_{s \to t} \frac{f(t) - f(s)}{(s - t)f(s)f(t)} = \lim_{s \to t} \left( \frac{f(s) - f(t)}{(s - t)} \right) \left( \frac{-1}{f(s)f(t)} \right) = -\frac{f'(t)}{(f(t))^2} = -f'(t)(f(t))^{-2}.
\]

• In particular, taking \( f(t) = t^n \), for \( n \) a positive integer and \( t \neq 0 \), taking \( f(t) = t^n \) in the previous result, we obtain:

\[
g(t) = \frac{1}{t^n}, \quad g'(t) = -\frac{nt^{n-1}}{(t^n)^2} = -\frac{nt^{n-1}}{t^{2n}} = -\frac{n}{t^{n+1}}.
\]

So the derivative function of \( t^{-n} \) is \(-nt^{-n-1}\).

So for any integer \( n \) we have:

\[
(t^n)' = nt^{n-1}.
\]

– If \( n > 0 \), we proved this earlier.

– If \( n < 0 \), put \( m = -n \), then \( m > 0 \) and \((t^n)' = (t^{-m})' = -mt^{-m-1} = nt^{n-1} \), as required.

– If \( n = 0 \), then \( t^0 = 1 \) a constant and the derivative of any constant function is zero, as required.

Since we have considered all possibilities for the integer \( n \), we are done.

**Examples**

– The derivative of \( t^{-4} \) is \(-4t^{-5}\).

– The derivative of \( t^{100} \) is \(100t^{99}\).

– The derivative of \( t^{-100} \) is \(-100t^{-101}\).

– The derivative of \( g(t) = \frac{1}{t^{\frac{1}{2}}} \) is \(-f'(t)(f(t))^{-2} \), where \( g(t) = \frac{1}{f(t)} \), so \( f(t) = t^{\frac{1}{2}} \).

So the derivative of \( g(t) = \frac{1}{t^{\frac{1}{2}}} = t^{-\frac{1}{2}} \) is:

\[
g'(t) = -\frac{\frac{1}{2}t^{-\frac{1}{2}}}{(t^{\frac{1}{2}})^2} = -\frac{1}{2}t^{-\frac{3}{2}}.
\]
The geometrical interpretation of the derivative

If \( y = f(t) \), then the quantity \( \frac{\Delta y}{\Delta t} = \frac{f(s) - f(t)}{s - t} \) is the slope of the chord joining the two points \((s, f(s))\) and \((t, f(t))\) of the graph of the function \( f \).

In the limit as \( s \to t \), the limit of the chord slope \( \frac{\Delta y}{\Delta t} \), if it exists, is just \( f'(t) \) and is called the slope of the tangent line to the curve at the point \((t, f(t))\).

Then the line through that point with that slope is called the tangent to the curve at that point.

The limit is often written \( \frac{dy}{dt} \) or \( \frac{df}{dt} \) rather than \( f'(t) \).

Note that a geometrical tangent line can sometimes exist at a point \( t = a \) even if the function is not differentiable at \( t = a \); for example the tangent line to \( y = t^{\frac{1}{3}} \) at the origin is the vertical line \( x = 0 \), but the slope there is infinite.

Examples

- Since the slope of \( y = t^2 \) at \( t = 3 \) is \( (2t)_{t=3} = 6 \), the tangent line has slope 6 and goes through the point \((3, 9)\), so it has the equation:

  \[
  y - 9 = 6(x - 3) = 6x - 18,
  \]

  \[
  y = 6x - 9.
  \]

- Since the slope of \( y = \frac{1}{x} \) at \( t = 2 \) is \( (-2t^{-3})_{t=2} = -\frac{1}{4} \), the tangent line has slope \(-\frac{1}{4}\) and goes through the point \((2, \frac{1}{4})\), so it has the equation:

  \[
  y - \frac{1}{4} = \frac{1}{4}(x - 2),
  \]

  \[
  x - 4y = 1.
  \]

- Since the slope of \( y = t^{\frac{1}{3}} \) at \( t = 4 \) is \( \frac{1}{2t^\frac{2}{3}}_{t=4} = \frac{1}{4} \), the tangent line has slope \( \frac{1}{4} \) and goes through the point \((4, 2)\), so it has the equation:

  \[
  y - 2 = \frac{1}{4}(x - 4) = \frac{x}{4} - 1,
  \]

  \[
  4y - x = 4.
  \]
The linear approximation to a differentiable function

If at a point \( t = a \), the function \( y = f(t) \) has derivative \( f'(a) \), then if we define, for \( t \neq a \),
\[
\epsilon(t) = \frac{f(t) - f(a)}{t - a} - f'(a),
\]
we have \( \epsilon(t) \to 0 \) as \( t \to a \).

Then we have:
\[
\epsilon(t) = \frac{f(t) - f(a)}{t - a} - f'(a),
\]
\[
f(t) - f(a) - f'(a)(t - a) = \epsilon(t)(t - a),
\]
\[
f(t) = f(a) + f'(a)(t - a) + \epsilon(t)(t - a),
\]

Put \( f_1(t) = f(a) + f'(a)(t - a) \), so we have \( f(t) = f_1(t) + \epsilon(t)(t - a) \).
Then \( f_1(t) \) is called the linear approximation to \( f(t) \) valid near \( t = a \).
The size of the error in this approximation is \( |\epsilon(t)||t - a|\) which is the product of two small quantities, when \( t \) is near \( a \), so should be small.
Thus the values of \( f_1 \) can be used to estimate the values of \( f \) near \( t = a \).

Examples

- The linear approximation to \( y = f(t) = t^2 \) at \( t = 2 \) is, since \( f(2) = 2^2 = 4 \) and \( f'(2) = 2t \big|_{t=2} = 2(2) = 4 \):
\[
f_1(t) = f(2) + f'(2)(t - 2) = 4 + 4(t - 2) = 4t - 8.
\]
The exact value of \( f(2.1) \) is 4.41.
We have \( f_1(2.1) = 4(2.1) - 8 = 8.4 - 8 = 4.4 \).
The fractional error in the linear approximation has size:
\[
\frac{0.01}{4.41} = \frac{1}{441} < \frac{1}{500} \text{ or less than one quarter of one percent.}
\]

- The linear approximation to \( y = f(t) = t^{3/2} \) at \( t = 4 \) is, since \( f(2) = \sqrt{4} = 2 \) and \( f'(2) = \frac{1}{2\sqrt{t}} \big|_{t=4} = \frac{1}{2\sqrt{4}} = \frac{1}{2(2)} = \frac{1}{4} \):
\[
f_1(t) = f(4) + f'(4)(t - 4) = 2 + \frac{1}{4}(t - 2) = \frac{t}{4} + 1.
\]
The exact value of \( f(4.41) \) is 2.1.
We have \( f_1(4.41) = \frac{4.41}{4} + 1 = 2.1025 \).
The fractional error in the linear approximation has size:
\[
\frac{0.0025}{2.1} = \frac{25}{2100} < \frac{1}{800} \text{ or less than one eighth of one percent.}
\]
The linear approximation to \( y = f(t) = t^{100} \) at \( t = 1 \) is, since we have
\[
f(1) = 1 \quad \text{and} \quad f'(1) = 100t^{99}\big|_{t=1} = 100:
\]
\[
f_1(t) = f(1) + f'(1)(t-1) = 1 + 100(t-1) = 100t - 99.
\]
The exact value of \( f(1.01) \) is

\[
2.704813829421526093267194710807530833677938382781002776890201049117
1015143067392794394560143467445909733565137548356426831251928176683
2427980496322329650055217977882315938008175933291885667484249510001.
\]

We have \( f_1(1.01) = 101 - 99 = 2. \)

The error in the linear approximation here is large, because the curve \( y = t^{100} \) deviates markedly from a straight line in the interval \([1, 1.015]\) as can be seen by plotting the graph, using Maple.

The linear approximation to \( f(t) = t^n \) at \( t = 1 \) is, since we have \( f(1) = 1 \) and \( f'(1) = nt^{n-1}\big|_{t=1} = n: \)
\[
f_1(t) = f(1) + f'(1)(t-1) = 1 + n(t-1) = nt - n + 1.
\]

The linear approximation to \( f(t) = \frac{1}{t} \) at \( t = 5 \) is, since we have \( f(5) = \frac{1}{5} \) and \( f'(5) = -\frac{1}{t^2}\big|_{t=5} = -\frac{1}{25}: \)
\[
f_1(t) = f(5) + f'(5)(t-5) = \frac{1}{5} - \frac{1}{25}(t-5) = \frac{1}{25}(10 - t).
\]

Then \( f(4.8) = \frac{1}{48} \) is estimated by:
\[
\frac{1}{25}(10 - 4.8) = \frac{5.2}{25} = \frac{26}{125} = 0.208.
\]

The exact value is \( \frac{1}{48} = \frac{10}{25} = \frac{5}{24} = 0.2833 \).

The percentage error is \( 100\left(1 - \left(\frac{26}{125}\right)\right) = \frac{100}{625}(625 - 624) = \frac{4}{25} = 0.16. \)
The physical interpretation of the derivative

If \( t \) is a physical variable, then the quantity \( \frac{f(s) - f(t)}{s - t} \) is called the average rate of change of \( f \) with respect to \( t \).

Then the derivative \( f'(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t} \) is called the (instantaneous) rate of change of \( f \) with respect to time at \( t \).

Examples

- If \( f \) is the position of a particle moving along a straight line, then \( f'(t) \) is called the velocity and \( |f'(t)| \) the speed of the particle.

- If \( f \) is the velocity of a particle moving along a straight line, then \( f'(t) \) is called the acceleration.

- If \( f(x) \) is the value of the electric potential at a point \( x \) on a straight line, then \( -f'(x) \) is called the electric field at \( x \).

- If \( m(t) \) is the amount of a radioactive element at time \( t \), then \( \frac{m'(t)}{m(t)} \) is called the decay rate (this is a constant for ordinary, exponential, radioactive decay). It measures the probability that an individual atom will decay in unit time.

- If \( P(t) \) is the population of a biological material at time \( t \), then \( \frac{P'(t)}{P(t)} \) is called the growth rate (this is a constant for exponential growth). It measures the probability that an individual member of the population will reproduce in unit time.
The differential

If \( f(t) \) has derivative \( f'(t) \) then the differential of \( f \) is \( df = f'(t)dt \).
Here \( dt \) is regarded as an independent variable.

Examples

- \( d(t^2) = 2tdt \),
- \( d\left(\frac{t}{2}^2\right) = \frac{dt}{2t^2} \),
- \( d\left(\frac{1}{t^2}\right) = -\frac{2dt}{t^3} \).

If a small change in \( t, \) \( t \to t + \Delta t \) induces a small change \( f \to f + \Delta f \) in \( f \), then we have approximately, 
\[ \frac{\Delta f}{\Delta t} = f'(t), \]
so \( \Delta f = f'(t)\Delta t \), approximately.

So the change \( \Delta f \) in \( f \) at \( t = a \), under \( t \to a + \Delta t \) is obtained approximately by evaluating the differential \( f'(t)dt \) with \( t \to a \) and \( dt \to \Delta t \):
\[ \Delta f = df\big|_{t=a, \; dt=\Delta t}. \]

This approximation which is equivalent to the linear approximation becomes more and more accurate as \( \Delta t \) gets smaller.