Complex variables: Quiz 4 Solutions 7/11/5

Question 1

Write the equations of the circle of radius 5 center \((3, 4)\) and the line \(4x - 3y = 25\) in standard form and hence determine their images under the transformations \(z \rightarrow \frac{1}{z}\) and \(z \rightarrow \frac{1}{z+1}\).

Plot the curves and their images under the various transformations on one graph.

- The circle is:

\[
25 = |z - 3 - 4i|^2 = (z - 3 - 4i)(\overline{z} - 3 + 4i) = z\overline{z} - (3 + 4i)z - (3 - 4i)\overline{z} + 25,
\]

\[
0 = z\overline{z} - (3 - 4i)z - (3 + 4i)\overline{z}.
\]

If \(w = \frac{1}{z}\), then \(z = \frac{1}{w}\), so the image of the circle is given by the equation:

\[
0 = \frac{1}{w} - (3 - 4i)\frac{1}{w} - (3 + 4i)\frac{1}{w}
\]

\[
0 = 1 - (3 - 4i)w - (3 + 4i)w.
\]

Replacing \(w\) by \(z\), gives the image as the line \((3 - 4i)\overline{z} + (3 + 4i)z = 1\).

Writing \(z = x + iy\), with \(x\) and \(y\) real, gives the image as:

\[
(3 - 4i)(x - iy) + (3 + 4i)(x + iy) = 1,
\]

\[
6x - 8y = 1.
\]

This is the straight line of slope \(\frac{3}{4}\) through the point \((0, -\frac{1}{8})\).

- If instead \(w = \frac{1}{z+1}\), then \(z + 1 = \frac{1}{w}\), so \(z = \frac{1}{w} - 1 = \frac{1-w}{w}\).

The equation of the circle becomes:

\[
0 = \left(\frac{1-w}{w}\right) \left(\frac{1-\overline{w}}{\overline{w}}\right) - (3 + 4i) \left(\frac{1-w}{w}\right) - (3 - 4i) \left(\frac{1-\overline{w}}{\overline{w}}\right),
\]

\[
0 = (1-w)(1-\overline{w}) - (3+4i)(1-w)\overline{w} - (3-4i)(1-\overline{w})w
\]

\[
= 1 - w - \overline{w} + w\overline{w} - (3 + 4i)\overline{w} + (3 + 4i)w\overline{w} - (3 - 4i)w + (3 - 4i)w\overline{w}
\]

\[
= 7w\overline{w} - (4 + 4i)\overline{w} - (4 - 4i)w + 1.
\]
Replacing $w$ by $z$ and writing $z = x + iy$, with $x$ and $y$ real, gives the image as:

\[
0 = 7x^2 + 7y^2 - (4 + 4i)(x - iy) - (4 - 4i)(x + iy) + 1
\]

\[
= 7x^2 + 7y^2 - 8x - 8y + 1
\]

\[
0 = x^2 + y^2 - \frac{8}{7}x - \frac{8}{7}y + \frac{1}{7},
\]

\[
(x - \frac{4}{7})^2 + (y - \frac{4}{7})^2 = -\frac{1}{7} + \frac{32}{49} = \left(\frac{5}{7}\right)^2.
\]

So the image circle has center \(\left(\frac{4}{7}, \frac{4}{7}\right)\) and has radius \(\frac{5}{7}\).

- The line \(4x - 3y = 25\) is:

\[
0 = -25 + \frac{4}{2}(z + \bar{z}) - \frac{3}{2i}(z - \bar{z})
\]

\[
0 = (4 + 3i)z + (4 - 3i)\bar{z} - 50.
\]

If \(w = \frac{1}{z}\), then \(z = \frac{1}{w}\), so the image of the line is given by the equation:

\[
0 = (4 + 3i)\frac{1}{w} + (4 - 3i)\frac{1}{\bar{w}} - 50
\]

\[
0 = 50w\bar{w} - (4 + 3i)\bar{w} - (4 - 3i)w.
\]

Replacing \(w\) by \(z\) and writing \(z = x + iy\), with \(x\) and \(y\) real, gives the image as:

\[
0 = x^2 + y^2 - \frac{1}{50}(4 + 3i)(x - iy) - \frac{1}{50}(4 - 3i)(x + iy)
\]

\[
= x^2 + y^2 - \frac{4}{25}x - \frac{3}{25}y
\]

\[
= (x - \frac{2}{25})^2 + (y - \frac{3}{50})^2 - \frac{4}{625} - \frac{9}{2500}
\]

\[
= (x - \frac{2}{25})^2 + (y - \frac{3}{50})^2 - \frac{1}{100}.
\]

This is the circle, center \(\left(\frac{2}{25}, \frac{3}{50}\right)\) of radius \(\frac{1}{10}\).
• If instead $w = \frac{1}{z+1}$, then $z + 1 = \frac{1}{w}$, so $z = \frac{1}{w} - 1 = \frac{1-w}{w}$.

The equation of the line becomes:

$$0 = (4 + 3i)z + (4 - 3i)\overline{z} - 50$$

$$= (4 + 3i)\frac{1-w}{w} + (4 - 3i)\frac{1-\overline{w}}{\overline{w}} - 50,$$

$$0 = (4 + 3i)(1-w)\overline{w} + (4 - 3i)(1-\overline{w})w - 50w\overline{w},$$

$$0 = (4 + 3i)\overline{w} + (4 - 3i)w - 58w\overline{w}.$$ 

Replacing $w$ by $z$ and writing $z = x + iy$, with $x$ and $y$ real, gives the image as:

$$0 = 58(x^2 + y^2) + 8x + 6y,$$

$$0 = (x + \frac{2}{29})^2 + (y + \frac{3}{58})^2 - \frac{4}{841} - \frac{9}{3364}$$

$$0 = (x + \frac{2}{29})^2 + (y + \frac{3}{58})^2 - \frac{25}{3364}.$$ 

So the image circle has center $(-\frac{2}{29}, -\frac{3}{58})$ and has radius $\frac{5}{3364}$. 


Question 2

Compute the following limits, or explain why the limit in question does not exist.

- \[
\lim_{z \to -2i} \frac{z^2 + 4}{z + 2i}
\]

We have:

\[
\lim_{z \to -2i} \frac{z^2 + 4}{z + 2i} = \lim_{z \to -2i} \frac{(z + 2i)(z - 2i)}{z + 2i} = \lim_{z \to -2i} (z - 2i) = -2i - 2i = -4i.
\]

- \[
\lim_{z \to 2+i} \frac{z^2 - 3 - 4i}{z^2 - 4z + 5}
\]

We have:

\[
\lim_{z \to 2+i} \frac{z^2 - 3 - 4i}{z^2 - 4z + 5} = \lim_{z \to 2+i} \frac{(z - 2 - i)(z + 2 + i)}{(z - 2 + i)(z - 2 - i)} = \lim_{z \to 2+i} \frac{(z + 2 + i)}{(z - 2 + i)} = 1 - 2i.
\]

- \[
\lim_{z \to 0} \frac{\Re(z^2)}{z}
\]

We have \(|\frac{\Re(z^2)}{z}| = \frac{|\Re(z^2)|}{|z|} \leq \frac{|z|^2}{|z|} = |z| \to 0\), as \(z \to 0\), so the required limit exist and is 0.

Alternatively we write \(z = re^{it}\) with \(r > 0\) and \(t\) real.
Then \(z^2 = re^{2it}\) and \(\Re(z^2) = r^2 \cos(2t)\).
Then we have:

\[
\lim_{z \to 0} \frac{\Re(z^2)}{z} = \lim_{r \to 0^+} \frac{r^2 \cos(2t)}{re^{it}} = \lim_{r \to 0^+} re^{-it} \cos(2t) = 0.
\]

Here we need only note that \(|e^{-it} \cos(2t)| \leq 1\).
Question 3

Show that the function $u(x, y) = e^{2x} \sin(y) \cos(y)$ is harmonic.

Does a real function $v(x, y)$ exist such that $u(x, y) + iv(x, y)$ is analytic? Explain.

We have if $u = e^{2x} \sin(y) \cos(y)$,

\[
\begin{align*}
  u_x &= 2e^{2x} \sin(y) \cos(y), \\
  u_{xx} &= 4e^{2x} \sin(y) \cos(y), \\
  u_y &= e^{2x} (\cos^2(y) - \sin^2(y)), \\
  u_{yy} &= e^{2x} (-2 \sin(y) \cos(y) - 2 \sin(y) \cos(y)) \\
            &= -4e^{2x} \sin(y) \cos(y), \\
  u_{xx} + u_{yy} &= 4e^{2x} \sin(y) \cos(y) - 4e^{2x} \sin(y) \cos(y) = 0.
\end{align*}
\]

So $u$ is harmonic, as required.

The function $f(x, y) = u(x, y) + iv(x, y)$ will be analytic if the Cauchy-Riemann equations hold:

\[
\begin{align*}
  v_x &= -u_y, \\
  v_y &= u_x.
\end{align*}
\]

So here we need first:

\[
\begin{align*}
  v_x &= -e^{2x} (\cos^2(y) - \sin^2(y)),
\end{align*}
\]

Integrating both sides with respect to $x$ gives the relation:

\[
\begin{align*}
  v &= -\frac{1}{2} e^{2x} (\cos^2(y) - \sin^2(y)) + C(y).
\end{align*}
\]

Finally we need:

\[
\begin{align*}
  v_y &= -\frac{1}{2} e^{2x} (-4 \sin(y) \cos(y)) + C'(y) = 2e^{2x} \sin(y) \cos(y) + C'(y) = u_x = 2e^{2x} \sin(y) \cos(y).
\end{align*}
\]

This gives just $C'(y) = 0$, so $C(y)$ is a constant $A$.

So $v = \frac{1}{2} e^{2x} (\sin^2(y) - \cos^2(y)) + A$ makes $u + iv$ harmonic, as required.

One can also see this as follows:

\[
\begin{align*}
  u(x, y) = e^{2x} \sin(y) \cos(y) = \frac{1}{2} e^{2x} \sin(2y) = \Re \left( \frac{1}{2i} e^{2(x+iy)} \right)
\end{align*}
\]

So $v = \Im \left( \frac{1}{2i} e^{2(x+iy)} \right) = -\frac{1}{2} e^{2x} \cos(2y)$ will do.

Since $\cos(2y) = \cos^2(y) - \sin^2(y)$, this agrees with our previous result.