Complex variables: Quiz 2 Solutions
Tuesday June 28th 2005

Question 1

Sketch the following sets in the complex plane and for each identify whether the set is open, closed or neither and whether or not the set is connected.
For each of these sets also give a parametrization or parametrizations of its boundary, as appropriate.

• \(A = \{z : \Re(z) \geq \Im(z)\}\).

Putting \(z = x + iy\), we need \(x \geq y\). This is the closed connected region below and to the right of the line \(y = x\) in the \((x, y)\)-plane.
The boundary is is the line \(y = x\), which is parametrized by \(x = t\), \(y = t\), so \(z = t + it = t(1 + i)\) where \(t\) is an arbitrary real number.
Alternatively the boundary is given by \(z = s \exp(i\frac{\pi}{4})\), where \(s\) is any real number.

• \(B = \{z : |z| < 1 \text{ and } |z - i| < 1\}\).

This is the open connected region in the complex plane common to the open discs of radius one, one centered at the origin, the other centered at \([0, 1]\). The two circular boundaries of the discs (which are not part of the region \(B\)) meet where:

\[
\begin{align*}
|z| &= 1, \quad |z - i| = 1, \quad |z|^2 = 1, \quad |z - i|^2 = 1, \\
z \bar{z} &= 1, \quad 1 = (z - i)(\bar{z} + i) = z \bar{z} - iz + i + 1, \\
z \bar{z} &= 1, \quad -iz + iz + 1 = 0, \\
x^2 + y^2 &= 1, \quad -2y + 1 = 0, \quad y = \frac{1}{2}, \quad x = \pm \frac{\sqrt{3}}{2}.
\end{align*}
\]

- The upper boundary of the region lies on the circle \(|z| = 1\) going counterclockwise from \((\frac{\sqrt{3}}{2}, \frac{1}{2})\) to \((\frac{\sqrt{3}}{2}, -\frac{1}{2})\).
  It may be parametrized by \(z = \exp(it)\), where \(\frac{\pi}{6} \leq t \leq \frac{5\pi}{6}\).
- The lower boundary of the region lies on the circle \(|z - i| = 1\) going counterclockwise from \((\frac{\sqrt{3}}{2}, -\frac{1}{2})\) to \((\frac{\sqrt{3}}{2}, \frac{1}{2})\).
  It may be parametrized by \(z = i + \exp(is)\), where \(\frac{7\pi}{6} \leq s \leq \frac{11\pi}{6}\).
\[ C = \{ z = re^{i\theta} : 1 < r < 2 \text{ and } -\frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3} \}. \]

This region is neither closed nor open and is connected.

It is the part of the annulus centered at the origin of inner radius 1 and outer radius 2, bounded on the left by the radial line at angle \( \theta = \frac{2\pi}{3} \) and on the right by the radial line at angle \( \frac{\pi}{3} \).

The circular edges of the annulus are not part of \( C \).
The straight edges of the annulus are part of \( C \).
The corners of the annulus (none of which are in \( C \), taken counterclockwise around the boundary of the annulus are the points:
\[ A = \exp(i\frac{\pi}{3}), \quad B = 2\exp(i\frac{\pi}{3}), \quad C = 2\exp(2i\frac{\pi}{3}) \text{ and } D = 2\exp(i\frac{\pi}{3}). \]

- The edge \( AB \) is parametrized by:
  \[ z = s\exp(i\frac{\pi}{3}), \quad 1 \leq s \leq 2. \]

- The edge \( BC \) is parametrized by:
  \[ z = 2\exp(it\frac{\pi}{3}), \quad 1 \leq t \leq 2. \]

- The edge \( CD \) is parametrized by:
  \[ z = (2 - u)\exp(i\frac{2\pi}{3}), \quad 0 \leq u \leq 1. \]

- The edge \( DA \) is parametrized by:
  \[ z = \exp(i\frac{(2 - v)\pi}{3}), \quad 0 \leq v \leq 1. \]
Question 2

Find all complex numbers $x$ such that $x^6 = -64$ and plot the solutions on the complex plane.

- We need $x^6 = -64 = 2^6i^6$.
  Put $x = 2iy$, for some complex number $y$ ($y = \frac{z}{2^n}$).
  Then we need $x^6 = 2^6i^6y^6 = 2^6i^6$, so $y^6 = 1$, whose roots we know are
  $y = \exp(\frac{2\pi ik}{6})$, where $k$ is any integer.
  So the required equation has the six distinct roots:
  $x = 2i\exp(\frac{2\pi ik}{6})$, $k = 0, 1, 2, 3, 4, 5$.

- Alternatively we can factor:
  $x^6 + 64 = (x^3 + 8i)(x^3 - 8i) = 0$, if $x^3 = 8i$, or $x^3 = -8i$.
  Noting that $(2i)^3 = 2^3i^3 = -8i$, we see that one root of $x^3 = 8i$ is $-2i$
  and one root of $x^3 = -8i$ is $2i$.
  Then the roots of $x^3 = 8i$ are $-2i, -2i\omega$ and $-2i\omega^2$ and the roots of
  $x^3 = 8i$ are $2i, 2i\omega$ and $2i\omega^2$, where $\omega$ is a cube root of unity that is not
  $1$ (so $\omega^2 + \omega + 1 = 0$).

- Alternatively, we use the polar decomposition:
  $x = r\exp(i\theta)$, $r > 0$, $\theta$ real.
  Then we need:
  $x^6 = (r\exp(i\theta))^6 = r^6\exp(6i\theta) = -64 = 64\exp(i\pi)$.
  So $r^6 = 64$, so $r = 2$ and $6\theta = \pi + 2k\pi$, where $k$ is an integer.
So the roots are:

\[ x = 2 \exp \left( \frac{i(2k + 1)\pi}{6} \right), \ k = 0, 1, 2, 3, 4, 5. \]

- When \( k = 0 \) we get:

\[ x = 2 \exp \left( \frac{i\pi}{6} \right) = 2 \left( \cos \left( \frac{\pi}{6} \right) + i \sin \left( \frac{\pi}{6} \right) \right) = \sqrt{3} + i. \]

- When \( k = 1 \) we get:

\[ x = 2 \exp \left( \frac{3i\pi}{6} \right) = 2 \left( \cos \left( \frac{\pi}{2} \right) + i \sin \left( \frac{\pi}{2} \right) \right) = 2i. \]

- When \( k = 2 \) we get:

\[ x = 2 \exp \left( \frac{5i\pi}{6} \right) = 2 \left( \cos \left( \frac{5\pi}{6} \right) + i \sin \left( \frac{5\pi}{6} \right) \right) = -\sqrt{3} + i. \]

- When \( k = 3 \) we get:

\[ x = 2 \exp \left( \frac{7i\pi}{6} \right) = 2 \left( \cos \left( \frac{7\pi}{6} \right) + i \sin \left( \frac{7\pi}{6} \right) \right) = -\sqrt{3} - i. \]

- When \( k = 4 \) we get:

\[ x = 2 \exp \left( \frac{9i\pi}{6} \right) = 2 \left( \cos \left( \frac{3\pi}{2} \right) + i \sin \left( \frac{3\pi}{2} \right) \right) = -2i. \]

- When \( k = 5 \) we get:

\[ x = 2 \exp \left( \frac{11i\pi}{6} \right) = 2 \left( \cos \left( \frac{11\pi}{6} \right) + i \sin \left( \frac{11\pi}{6} \right) \right) = \sqrt{3} - i. \]

So the roots are \( 2i, -2i, \sqrt{3} - i, \sqrt{3} + i, -\sqrt{3} + i \) and \( \sqrt{3} - i \).

The six roots taken together form a regular hexagon in the complex plane inscribed in the circle of radius two, center the origin, with two horizontal parallel sides.
Question 3

Suppose that a real polynomial has a factor of \(x - \alpha\), where \(\alpha\) is a complex number that is not real.

Prove that the polynomial has a real factor of the form \(x^2 - 2\Re(\alpha)x + |\alpha|^2\).

Verify that \(x = i\) is a root of the polynomial \(x^4 - 4x^3 + 6x^2 - 4x + 5 = 0\) and hence find all its roots.

If the real polynomial in question is \(p(x)\) and has a factor of \((x - \alpha)\), then it has a root \(x = \alpha\), so we have \(p(\alpha) = 0\). Taking the complex conjugate, since the coefficients of the polynomial are real, we get \(p(\overline{\alpha}) = 0\).

Now since the polynomial has a factor of \((x - \alpha)\), we have \(p(x) = (x - \alpha)q(x)\) for some complex polynomial \(q(x)\). Putting \(x = \overline{\alpha}\), we get:

\[
0 = p(\overline{\alpha}) = (\overline{\alpha} - \alpha)q(\overline{\alpha}).
\]

But since \(\alpha\) is not real, we have \(\overline{\alpha} - \alpha = -2i\Im(\alpha) \neq 0\), so we must have \(q(\overline{\alpha}) = 0\), so \(q(x)\) has a root \(x = \overline{\alpha}\) and so \(q(x)\) factorizes as \(q(x) = (x - \overline{\alpha})r(x)\), where \(r(x)\) is a polynomial. Then we get:

\[
p(x) = (x - \alpha)q(x) = (x - \alpha)(x - \overline{\alpha})r(x)
= (x^2 - (\alpha + \overline{\alpha})x + \alpha\overline{\alpha})r(x) = (x^2 - 2\Re(\alpha)x + |\alpha|^2)r(x).
\]

So the polynomial \(p(x)\) has a real factor of \(x^2 - 2\Re(\alpha)x + |\alpha|^2\), as required.

Note that by polynomial division it follows that the other factor \(r(x)\) is also a real polynomial.

Now put \(p(x) = x^4 - 4x^3 + 6x^2 - 4x + 5 = 0\).

Then we have \(p(i) = i^4 - 4i^3 + 6i^2 - 4i + 5 = 1 + 4i - 6 - 4i + 5 = 0\), so \(i\) is a root of \(p(x)\), so \(p(x)\) has a factor of \((x - i)\).

By the above argument, since the root is non-real and since \(p(x)\) is real, \(p(x)\) has a factor of \(x^2 - 2\Re(i) + |i|^2 = x^2 + 1\).

Now \(-4x^3 - 4x = -4x(x^2 + 1)\) and \(x^4 + 6x^2 + 5 = (x^2 + 1)(x^2 + 5)\), so:

\[
x^4 - 4x^3 + 6x^2 - 4x + 5 = (x^2 + 1)(-4x) + (x^2 + 1)(x^2 + 5) = (x^2 + 1)(x^2 - 4x + 5).
\]

Then the roots of \(p(x)\) are \(x = \pm i\) together with the roots of the polynomial \(x^2 - 4x + 5 = x^2 - 4x + 4 + 1 = (x - 2)^2 + 1 = (x - 2)^2 - (i)^2 = (x - 2 - i)(x - 2 + i)\).

So the roots are \(i, -i, 2 + i\) and \(2 - i\); the corresponding factorization is:

\[
x^4 - 4x^3 + 6x^2 - 4x + 5 = (x^2 + 1)(x^2 - 4x + 5) = (x - i)(x + i)(x - 2 - i)(x - 2 + i).
\]