Complex variables: Quiz 1 Solutions June 22nd 2005

Question 1

Let $a = 3 + 4i$ and $b = 1 - 2i$.

- Plot $a$, $b$, $ib$, $a - ib$ and $a + ib$ on the complex plane.

  We plot the points $a = (3, 4)$, $b = (1, -2)$, $ib = (2, 1)$, $a - ib = (1, 3)$ and $a + ib = (5, 5)$.

- Compute the sizes (moduli) of the complex numbers $a$, $b$, $ib$, $ab$, $a^2$ and $\frac{1}{a}$ and discuss your results.

  $a = 3 + 4i,$
  
  $b = 1 - 2i,$
  
  $ab = 11 - 2i,$
  
  $a^2 = 6 - 16 + 24i = -10 + 24i,$
  
  $\frac{1}{a} = \frac{\bar{a}}{|a|^2} = \frac{3}{25} - \frac{4}{25}i,$
  
  $|a| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5,$
  
  $|b| = \sqrt{1^2 + 2^2} = \sqrt{5},$
  
  $|ib| = \sqrt{2^2 + 1^2} = \sqrt{5},$
  
  $|ab| = \sqrt{11^2 + 2^2} = \sqrt{121 + 4} = \sqrt{125} = 5\sqrt{5},$
  
  $\frac{1}{|a|} = \sqrt{\left(\frac{3}{25}\right)^2 + \left(\frac{4}{25}\right)^2} = \sqrt{\frac{9}{625} + \frac{16}{625}} = \sqrt{\frac{25}{625}} = \sqrt{\frac{1}{25}} = \frac{1}{5}.$

We notice that multiplication by $i$ applied to $b$ does not change the modulus and the modulus of $ab$ is the product of the moduli of $a$ and $b$ and the modulus of the inverse of $a$ is the inverse of the modulus of $a$.

These relations all reflect the identity $|zw| = |z||w|$ which is true for all complex numbers $z$ and $w$. 
Question 2

The complex number \( z \) obeys the equation \((3 - i)z + (2 + i)\bar{z} = 4 - 3i\). What is \( z \)?

We write out the given equation in terms of real and imaginary parts. We put \( z = x + iy \), with \( x \) and \( y \) real. Then we need:

\[
4 - 3i = (3 - i)z + (2 + i)\bar{z} = (3 - i)(x + iy) + (2 + i)(x - iy) = 3x + y + i(3y - x) + 2x + y + i(x - 2y) = 5x + 2y + iy.
\]

Comparing real and imaginary parts we get the equations:

- \( 5x + 2y = 4 \), \( y = -3 \)
- \( 5x - 6 = 4 \), \( 5x = 10 \), \( x = 2 \)

So the solution is \( (x, y) = (2, -3) \) and \( z = 2 - 3i \).

Check:

\((3 - i)z + (2 + i)\bar{z} = (3 - i)(2 - 3i) + (2 + i)(2 + 3i) = 3 - 11i + 1 + 8i = 4 - 3i\).

Alternatively we write the equation and its complex conjugate as a linear system \( AZ = B \), where we have:

\[
A = \begin{bmatrix} 3 - i & 2 + i \\ 2 - i & 3 + i \end{bmatrix}, \quad Z = \begin{bmatrix} z \\ \bar{z} \end{bmatrix}, \quad B = \begin{bmatrix} 4 - 3i \\ 4 + 3i \end{bmatrix}.
\]

The solution is then:

\[
Z = A^{-1}B = \frac{1}{5} \begin{bmatrix} 3 + i & -2 - i \\ -2 + i & 3 - i \end{bmatrix} \begin{bmatrix} 4 - 3i \\ 4 + 3i \end{bmatrix} = \frac{1}{5} \begin{bmatrix} (3 + i)(4 - 3i) + (-2 - i)(4 + 3i) \\ (-2 + i)(4 - 3i) + (3 - i)(4 + 3i) \end{bmatrix} = \begin{bmatrix} 2 - 3i \\ 2 + 3i \end{bmatrix}.
\]

This gives the same solution as before.

Here we used that the inverse of a two by two matrix of non-zero determinant is given as follows:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
\]

Also the determinant of \( A \) here is \((3 + i)(3 - i) - (2 - i)(2 + i) = 10 - 5 = 5\).
Question 3

Sketch the circles in the complex plane with equations $|z + 2 - i| = 5$ and $|z| = 2\sqrt{5}$ and find their points of intersection.

The first circle is centered at $(-2, 1)$ and has radius 5.
The second circle is the circle of radius $2\sqrt{5}$, centered at the origin.

For the intersection, we first write $z = x + iy$, with $x$ and $y$ real.
Then we need:

\[
(2\sqrt{5})^2 = 20 = |z|^2 = x^2 + y^2,
\]
\[
25 = |z + 2 - i|^2 = |(x + 2) + i(y - 1)|^2,
\]
\[
25 = (x + 2)^2 + (y - 1)^2,
\]
\[
25 = x^2 + y^2 + 4x - 2y + 5 = 25 + 4x - 2y,
\]
\[
4x - 2y = 0,
\]
\[
y = 2x,
\]
\[
5x^2 = 20,
\]
\[
(x, y) = (2, 4) \text{ or } (x, y) = (-2, -4)
\]
\[
z = 2 + 4i \text{ or } z = -2 - 4i.
\]

We check:
\[
|2 + 4i| = |-2 - 4i| = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5},
\]
\[
|2 + 4i + 2 - i| = |4 + 3i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5,
\]
\[
|-2 - 4i + 2 - i| = |-5i| = \sqrt{0^2 + 5^2} = \sqrt{25} = 5.
\]
So both points lie on both circles, as required.