Complex Variables, Summer 2005
Homework Assignments
Homework 1, due for discussion Tuesday June 21st and to be turned in Wednesday June 22nd

• Let \( a = 2 + 3i \) and \( b = 4 + 5i \).
  Sketch the complex numbers \( a, b, a^2, ab, ba, b^2, \frac{a}{b} \) and \( \frac{b}{a} \).
  For each determine their size (modulus) and determine the angle that the vector from the origin to the complex number in question makes with the positive \( x \)-axis.
  Discuss your results.

• Find the solutions of the equations \( x^3 = 1 \) and \( x^6 = 1 \) and plot the solutions in the complex plane.
  Discuss your results.

• A complex number \( z \) obeys the relations \( |z - i| = 3\sqrt{2} \) and \( |z + 3| = 4 \).
  What can we say about the number \( z \)?
  In particular is \( z \) unique?
  Explain your answer graphically.
Homework 2, due for discussion Monday June 27th and
to be turned in Tuesday June 28th

- Let \( z, w \) and \( t \) be complex numbers representing points \( A, B \) and \( C \),
in the complex plane, respectively.
  Show that the complex number \( \frac{1}{2}(z + w) \) represents the midpoint of the
  segment \( AB \) and that the complex number \( \frac{1}{3}(z + w + t) \) represents the
  centroid of the triangle \( ABC \).

- Prove that \( |z - w|^2 = |z|^2 + |w|^2 - 2\Re(z\overline{w}) \), for any complex numbers
  \( z \) and \( w \).

- Let \( \omega \) be a cube root of 1 that is not 1 itself (so \( \omega^3 = 1 \), but \( \omega \neq 1 \)).
  Show that \( \omega^2 - \omega + 1 = 0 \).
  Show that \( \omega^2 \) is also a cube root of 1 that is not 1.
  Also express \( \frac{\omega + 1}{\omega - 1} \) as a linear combination of \( \omega \) and 1 with real coe-
  cients.

- Write \( a = \sqrt{3} - i \) and \( b = 1 + i \) in polar form.
  Hence obtain the polar representations of \( a^2 \), \( b^2 \) and \( \frac{a}{b} \) and verify that
  your answers are correct, by comparing with the direct evaluation of
  these quantities.
  Find the smallest positive integer \( n \) such that \( a^n \) and \( b^n \) are both real
  and positive.
  Also illustrate your results in the complex plane.

- Find all square roots of \( 3 + 4i \) and illustrate your results in the complex
  plane.
Homework 3, due Tuesday July 5th

- Let \( f(z) = \frac{z + 2 - i}{z - 1 + i} \).
  Give the domain of \( f(z) \) and write \( f(z) \) as \( u + iv \), where \( u \) and \( v \) are functions of the real variables \( x \) and \( y \), with \( z = x + iy \).

- The electric field at a point \( z \) of the complex plane, due to a charged line carrying charge \( q \), perpendicular to the plane and passing through the plane at point \( w \) is \( \frac{q}{z - w} \), where \( z \neq w \).
  The total electric field due to an assemblage of such charged lines is the sum of the individual fields.
  One line of charge \( q \) passes through the origin, and two others, each carrying charge \( 2q \) pass through the points \( 2 + i \) and \( 2 - i \).
  Where is the total field zero?

- Let \( w = (1 - i)z + 1 - 2i \), defined for any complex \( z \).
  Find the image in the \( w \)-plane of the following sets:
  - \( \{ z : \Im(z) > 1 \} \)
  - \( \{ z : \Re(z) = \Im(z) \} \)
  - \( \{ z : |z| = 1 \} \).

- Let \( w = z^2 \), defined for any complex \( z \).
  Find the image in the \( w \)-plane of the following sets:
  - \( \{ z : \Im(z) > 1 \} \)
  - \( \{ z : 1 < |z| < 2 \text{ and } -\frac{\pi}{3} \leq \Arg(z) < \frac{\pi}{2} \} \)
  - \( \{ z : z^5 = 1 \} \).

- Let \( w = \frac{1}{z} \), defined for any complex \( z \neq 0 \).
  Find the image of the following sets:
  - \( \{ z : |z| = 1 \} \)
  - \( \{ z : |z| = 2 \} \)
  - \( \{ z : \Re(z) = 1 \} \)
  - \( \{ z : \Re(z) + \Im(z) = 1 \} \)
  - \( \{ z : |z - i| = 1 \} \).
Homework 4, due Tuesday July 12th

- Find the image under the transformation $z \mapsto \frac{z+1}{z+2}$ of the circle center $2+2i$, radius 2 and of the straightline $z = 1 - it$.

- Let $f(x, y) = \frac{2xy}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$. Does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist? Explain.

- Let $f(z) = \frac{z^2}{|z|^2}$, for any complex $z \neq 0$. Find the limit of $f(z)$ as $z$ goes to zero along each of the following curves:
  - The line $y = x$.
  - Along the line $y = 2x$.
  - Along the parabola $y = x^2$.

  What can you conclude about the limit of $f(z)$ as $z \to 0$?

- Find the following limits:

  - $\lim_{z \to i} \frac{z^4 - 1}{z - i}$.

  - $\lim_{z \to -i} \frac{z^6 + 1}{z + i}$.

  - $\lim_{z \to 1+i} \frac{z^4 + 4}{z^2 - 2z + 2}$.

- Show that $f(z) = e^{-y}(\cos(x) + i \sin(x))$ obeys the Cauchy-Riemann equations.

- The function $x^3 - 3x^2 - 3xy^2 + 3ix^2y - 6ixy + g(y)$ obeys the Cauchy-Riemann equations. What is the function $g(y)$? Explain.
Homework 5, due Thursday July 14th

- Show that the function \( f(z) = x^3 + 3xy^2 + i(y^3 + 3x^2y) \) is complex differentiable on the \( x \) and \( y \) axes, but is analytic nowhere.

- Show that the function \( f(z) = \frac{1}{z} \), defined for all non-zero complex numbers \( z \), is analytic everywhere in its domain of definition.

- Show that the function \( u(x, y) = e^{-x} \cos(y) \) is harmonic and determine, with proof a function \( v(x, y) \) such that \( f(z) = u(x, y) + iv(x, y) \) is analytic.

- Prove that the functions \( \arctan\left(\frac{y}{x}\right) \) and \( \ln\left(\sqrt{x^2 + y^2}\right) \) are harmonic. Are these functions essentially harmonic conjugates of each other? Explain.

- Show that the function \( f(z) = \frac{\overline{z}^2}{z} \), for \( z \neq 0 \) and \( f(0) = 0 \) obeys the Cauchy-Riemann equations at the origin, but is nowhere differentiable.
Homework 6, due Tuesday July 19th

- Does \( \lim_{n \to \infty} \left( \frac{1 + i}{\sqrt{2}} \right)^n \) exist? Explain.

- Show that \( \sum_{n=0}^{\infty} \left( \frac{1}{n + 1 + i} - \frac{1}{n + i} \right) = i. \)

- Prove that if \( S = \sum_{n=0}^{\infty} |z_n| \) converges, then so does \( T = \sum_{n=0}^{\infty} z_n \) and then we have \( |T| \leq S. \)

- Show that each of the following series converges and determine its sum, if possible:

  - \( - \sum_{n=0}^{\infty} \left( \frac{1}{2 + i} \right)^n \)
  - \( - \sum_{n=0}^{\infty} \left( \frac{1 + i}{2} \right)^n \)
  - \( - \sum_{n=1}^{\infty} \frac{(1 + i)^n}{n(2 + i)^n} \)
  - \( - \sum_{n=1}^{\infty} \frac{(1 + i)^n}{n!} \)
  - \( - \sum_{n=0}^{\infty} \frac{(1 + i)^n}{(2n + 1)!} \)

- Find the disc of convergence of the following geometric series and sum each series in its disc of convergence:

  - \( - \sum_{n=0}^{\infty} (1 + i)^n z^n \)
  - \( - \sum_{n=0}^{\infty} \frac{z^n}{(1 + 2i)^{2n}} \)
  - \( - \sum_{n=0}^{\infty} 2^{-n}(z - 3 - 4i)^n \)
Homework 7, due Thursday July 21st

- Determine the disc of convergence of each of the following series and give a simple formula for its sum, if possible.

\[ \sum_{n=0}^{\infty} \frac{(z - i)^n}{2^n} \]
\[ \sum_{n=0}^{\infty} \frac{(z - 3)^n}{n!} \]
\[ \sum_{n=0}^{\infty} (2^n + 3^n)z^n \]
\[ \sum_{n=0}^{\infty} \frac{n(z - 1)^n}{(n + 1)^2} \]

- Let \( A(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^n(n + 1)} \).

  - Determine the convergence disc of \( A(z) \).
  - Give formulas for the first two derivatives of \( A(z) \). Do they converge for the same \( z \)-values as does \( A(z) \)? Explain.
  - Give a formula for the series \( B(z) \), such that \( B'(z) = A(z) \) and \( B(0) = 0 \). Does the series \( B(z) \) converge for the same \( z \)-values as does \( A(z) \)? Explain.
  - Can you give a simple expression for the sum \( A(z) \)?
Homework 8, due Wednesday July 27th

- Compute \( \ln(-2+3i) + \ln(1+4i) - \ln((-2+3i)(1+4i)) \) and explain graphically why the result is non-zero.

- Find all values of \((1 + i)^{2-3i}\) and \(i^{\frac{3}{2}}\).

- Compute the integrals:
  \[
  \int_{0}^{2} \frac{t}{t+i} dt, \\
  \int_{0}^{\infty} e^{-zt} dt, \text{ where } \Re(z) > 0.
  \]

- Evaluate the integrals \(\int_{\Gamma_1} zdz\) and \(\int_{\Gamma_2} zdz\), where \(\Gamma_1\) is the upper half of the circle in the complex plane centered at the origin of radius 1, traced counter-clockwise and \(\Gamma_2\) is the polygonal path starting at \(z = 1\) going first in a straight line to \(z = 1+i\), then in a straight line to \(z = i\), then in a straight line to \(z = -1\).

- Determine the integrals around the unit circle centered at the origin, traced once, counterclockwise, of the following functions:
  - \(f_1(z) = \frac{z}{z^2+2}\)
  - \(f_2(z) = \frac{1}{z^2+2z+2}\)
  - \(f_3(z) = \frac{1}{4z^2+1}\)

- Find the integral of \(z^{-1}(z-2)^{-1} \exp(z)\) around the circle center the origin radius \(\frac{1}{2}\) traced once counterclockwise.

  If instead the circle has radius \(r\) center the origin, which values of \(r\) will give a result for the integral of zero?