Topics in geometry Quiz 6 Solutions 7/26/5

Question 1

Consider the following Euclidean transformation matrix $\mathcal{M}$:

$$
\mathcal{M} = \begin{bmatrix}
\frac{4}{5} & -\frac{3}{5} & -2 \\
\frac{4}{5} & \frac{3}{5} & 2 \\
0 & 0 & 1
\end{bmatrix}
$$

- Describe the transformation $\mathcal{M}$ geometrically.

We recognize the standard form of a direct Euclidean transformation, which is clearly a rotation, since it is not a translation.

The angle of rotation is counterclockwise through an angle $\theta$, where we have $\cos(\theta) = \frac{4}{5}$ and $\sin(\theta) = \frac{3}{5}$.

Then we have $\tan(\theta) = \frac{3}{4}$, so $\theta = \arctan(\frac{3}{4}) = 0.6435011088$ radians, or $36.86989765$ degrees. The rotation center is given by the solution of the following equation:

$$
\mathcal{M}X = X,
$$

$$
\begin{bmatrix}
\frac{4}{5} & -\frac{3}{5} & -2 \\
\frac{4}{5} & \frac{3}{5} & 2 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix} =
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix}.
$$

$$
\frac{4}{5}x - \frac{3}{5}y - 2 = x, \quad \frac{3}{5}x + \frac{4}{5}y + 2 = y,
$$

$$
4x - 3y - 10 = 5x, \quad 3x + 4y + 10 = 5y,
$$

$$
-3y - 10 = x, \quad 3x - y + 10 = 0,
$$

$$
-3y - 10 = x, \quad 3(-3y - 10) - y + 10 = 0,
$$

$$
-3y - 10 = x, \quad -10y - 20 = 0,
$$

$$
y = -2, \quad x = -3(-2) - 10 = -4.
$$

So the rotation is a counter-clockwise rotation through $36.86989765$ degrees, about the center $(-4, -2)$. 
• Find its fixed points and fixed lines, if any.

Since this is a rotation, through an angle which is not an integer multiple of \( \pi \), there are no fixed lines and the only invariant point is the center of rotation, which is the point \((-4, -2)\).

• Find the image of the line \(2x - 3y + 1 = 0\) under the action of the transformation \( \mathcal{M} \).

Three methods:

– First we know that \( \mathcal{M}^{-1} \) is a rotation through \(-\arctan\left(\frac{3}{4}\right)\), so takes the following form:

\[
\mathcal{M}^{-1} = \begin{bmatrix}
\frac{4}{5} & \frac{3}{5} & a \\
-\frac{3}{5} & \frac{4}{5} & b \\
0 & 0 & 1
\end{bmatrix}
\]

The point \(Y = (-4, -2)\) must be invariant, which gives the equation:

\[
\mathcal{M}^{-1} Y = Y,
\]

\[
\begin{bmatrix}
\frac{4}{5} & \frac{3}{5} & a \\
-\frac{3}{5} & \frac{4}{5} & b \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-4 \\
-2 \\
1
\end{bmatrix}
= 
\begin{bmatrix}
-4 \\
-2 \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
-16/5 \\
-6/5 \\
1
\end{bmatrix}
+ a = -4,
\]

\[
b = -2,
\]

\[
a = \frac{22}{5} - 4 = \frac{2}{5},
\]

\[
b = -\frac{4}{5} - 2 = -\frac{14}{5},
\]

\[
\mathcal{M}^{-1} = \begin{bmatrix}
\frac{4}{5} & \frac{3}{5} & \frac{2}{5} \\
-\frac{3}{5} & \frac{4}{5} & -\frac{14}{5} \\
0 & 0 & 1
\end{bmatrix}
\]

Then the line \(2x - 3y + 1 = 0\) has line co-ordinates \([2, -3, 1]\), so its image is:

\[
[2, -3, 1]\mathcal{M}^{-1} = [2, -3, 1]
\begin{bmatrix}
\frac{4}{5} & \frac{3}{5} & \frac{2}{5} \\
-\frac{3}{5} & \frac{4}{5} & -\frac{14}{5} \\
0 & 0 & 1
\end{bmatrix}
= [\frac{17}{5}, -\frac{6}{5}, \frac{51}{5}].
\]

So the image line has the equation \(17x - 6y + 51 = 0\).
We pick two points of the line and find their images under the action of $M$.

The required line is then the line through the images.

We pick $(x, y) = (1, 1)$ and $(x, y) = (-2, -1)$, which both lie on $2x - 3y + 1 = 0$, as is easily checked, giving the images:

$$
\begin{bmatrix}
\frac{4}{5} & -\frac{3}{5} & -2 \\
\frac{3}{5} & \frac{4}{5} & 2 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & -2 \\
1 & -1 \\
1 & 1
\end{bmatrix}
= \begin{bmatrix}
-\frac{9}{5} & -3 \\
\frac{17}{5} & 0 \\
0 & 0
\end{bmatrix}
$$

The image line goes through $(-3, 0)$ and $(-\frac{9}{5}, \frac{17}{5})$.

Then its slope is $\frac{\frac{17}{5}}{-\frac{9}{5} + 3} = \frac{17}{-9 + 15} = \frac{17}{6}$, so has the image line has the equation:

$$y - 0 = \frac{17}{6}(x - (-3)),
6y = 17x + 51,
17x - 6y + 51 = 0.$$

We parametrize the line $(x, y) = (1, 1) + t(3, 2) = (1 + 3t, 1 + 2t)$ and apply $M$:

$$
\begin{bmatrix}
\frac{4}{5} & -\frac{3}{5} & -2 \\
\frac{3}{5} & \frac{4}{5} & 2 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 + 3t \\
1 + 2t \\
1
\end{bmatrix}
= \begin{bmatrix}
-\frac{9}{5} + \frac{6t}{5} \\
\frac{17}{5} + \frac{12t}{5} \\
1
\end{bmatrix}
$$

So the image line has the parametric equations:

$$x = \frac{1}{5}(-9 + 6t),
\quad y = \frac{1}{5}(17 + 17t).$$

Eliminating $t$, we get:

$$17x - 6y = \frac{1}{5}(-9(17) - 6(17)) = \frac{1}{5}(-15(17)) = -3(17) = -51,$$

$$17x - 6y + 51 = 0.$$

In all cases, we get the same equation for the image line.
Question 2

Find the real matrix affine transformation $N$ which takes:

$A = (1, 1)$ to $A' = (6, 2),
B = (4, 1)$ to $B' = (6, -1),
C = (1, 5)$ to $C' = (2, 2)$.

- Is this transformation Euclidean?
  Explain.
- What is the transformation $N^2$?
- Also describe the transformation $N$ geometrically and determine its fixed points and lines, if any.

We have:

- $X = AB = B - A = (3, 0), X' = A'B' = B' - A' = (0, -3),$
- $Y = AC = C - A = (0, 4), Y' = A'C' = C' - A' = (-4, 0),$
- $Z = BC = C - B = (-3, 4), Z' = B'C' = C' - B' = (-4, 3).$

We see that $|AB| = |A'B'| = 3, |AC| = |A'C'| = 4$ and $|BC| = |B'C'| = 5$. Also $A, B$ and $C$ are not collinear, since $|AB| + |AC| \neq |BC|$, so the transformation is distance preserving and is therefore Euclidean.

The matrix $A$ representing the non-translational part of the transformation is determined by the formula:

$$A = \begin{bmatrix} 0 & -4 \\ -3 & 0 \end{bmatrix}.$$ 

Then the transformation is $x \rightarrow Ax + v$, for some vector $v$. 

Applying the transformation to the vector $A$, we get:

\[ A' = A\bar{A} + v. \]

\[ v = -A\bar{A} + A' \]

\[
\begin{vmatrix}
0 & -1 & 1 \\
-1 & 0 & 1
\end{vmatrix}
+ 
\begin{vmatrix}
6 & -1 \\
2 & -1
\end{vmatrix}
= 
\begin{vmatrix}
7 \\
3
\end{vmatrix}
\]

The $3 \times 3$ matrix $N$ representing the transformation is then:

\[
N = \begin{vmatrix}
0 & -1 & 7 \\
-1 & 0 & 3 \\
0 & 0 & 1
\end{vmatrix}
\]

We see easily that $\det(N) = -1$ and $N^2 \neq I$, so $N$ is a glide reflection.

Computing $N^2$ we get:

\[
N^2 = \begin{vmatrix}
0 & -1 & 7 \\
-1 & 0 & 3 \\
0 & 0 & 1
\end{vmatrix}
\begin{vmatrix}
0 & -1 & 7 \\
-1 & 0 & 3 \\
0 & 0 & 1
\end{vmatrix}
= \begin{vmatrix}
1 & 0 & 4 \\
0 & 1 & -4 \\
0 & 0 & 1
\end{vmatrix}
\]

So $N^2$ is the non-zero translation $(x, y) \rightarrow (x + 4, y - 4)$, confirming that $N$ is a glide reflection, the mirror of the reflection being parallel to the line $y = -x$.

In particular it has no fixed points and a unique invariant line, the line $L$ of reflection.

Then $L$ goes through the midpoints $P$ and $Q$ of the segments $AA'$ and $BB'$, respectively.

We have:

\[
P = \frac{1}{2}(A + A') = \left(\frac{7}{2}, \frac{3}{2}\right),
\]

\[
Q = \frac{1}{2}(B + B') = (5, 0).
\]

So the line $PQ$ has slope $\frac{0 - 3}{5 - \frac{7}{2}} = -1$.

By the point slope form of the equation of a line, the line $L$ has the equation:

\[ y = 0 - (x - 5), \quad x + y - 5 = 0. \]

So the line with line co-ordinates $L = (1, 1, -5)$ is invariant.

It is easily checked that $LN = (1, 1, -5)N = -(1, 1, -5) = -L$, as expected.
Note that the point $B = (4, 1)$ lies on this line, so it is invariant under the reflection in the line.

To get from the point $B$ to $B' = (6, -1)$ using a translation requires a translation through $B' - B = (2, -2)$, so the glide reflection consists of a reflection in the line $L$ followed by a translation through $(2, -2)$, which is a translation down to the right, parallel to the line $L$, through a distance $2\sqrt{2}$.

Alternatively, we can proceed by first drawing the triangles, so we see that an indirect transformation is needed, so next we locate the mirror, using the midpoints $P$ and $Q$, as above.

Then we see that the mirror has slope $t = -1$, so the required (indirect) transformation has the form:

$$
\mathcal{N} = \begin{pmatrix}
\frac{1-t^2}{1+t^2} & \frac{2t}{1+t^2} & a \\
\frac{2t}{1+t^2} & \frac{t^2-1}{1+t^2} & b \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
0 & -1 & a \\
-1 & 0 & b \\
0 & 0 & 1
\end{pmatrix}.
$$

Then $a$ and $b$ are determined by the action on $A$ (say):

$$
\mathcal{N}A = A',
$$

$$
\begin{pmatrix}
0 & -1 & a \\
-1 & 0 & b \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}1
\end{pmatrix} = \begin{pmatrix}6
\end{pmatrix},
$$

$$
a - 1 = 6, \quad b - 1 = 2,
$$

$$
a = 7, \quad b = 3,
$$

$$
\mathcal{N} = \begin{pmatrix}
0 & -1 & 7 \\
-1 & 0 & 3 \\
0 & 0 & 1
\end{pmatrix}.
$$