Topics in Geometry, Quiz 3 Solutions, 7/5/5

Question 1

Consider the $\mathbb{Z}_7$ projective geometry.

- Find the equation of the line $\mathcal{R}$ through the points $A = (1, 3, 6)$ and $(3, 1, 5)$.

If the line $\mathcal{R}$ is $ax + by + cz = 0$, we need $a + 3b + 6c = 0$ and $3a + b + 5c = 0$. The first equation gives $c = a + 3b$.

Replacing $c$ by $a + 3b$ in the second equation gives:

$$0 = 3a + b + 5(a + 3b) = 8a + 16b = a + 2b,$$

so $a = -2b$.

Then $c = a + 3b = -2b + 3b = b$.

So now we have $(a, b, c) = (-2b, b, b) = b(-2, 1, 1)$, so $b \neq 0$.

After scaling by $b^{-1}$, we may take the equation of the line to be:

$$0 = -2x + y + z, \text{ or } y + z = 2x.$$

We see that both points $A$ and $B$ obey this equation, as required.

- Determine the other points of the line $\mathcal{R}$.

  - If we put $z = 0$, then $y = 2x$, so the point is $(x, 2x, 0) = x(1, 2, 0)$.
    So the point at infinity of the line $\mathcal{R}$ is the point $P_\infty = (1, 2, 0)$.

  - If we put $z \neq 0$, then after scaling, we may assuming that $z = 1$, giving $y + 1 = 2x$, or $y = 2x + 6$.
    Then we vary $x$ to obtain the other points of $\mathcal{R}$:
      * Putting $x = 0$, gives $y = 6$ and the point $P_0 = (0, 6, 1)$ on $\mathcal{R}$.
      * Putting $x = 1$, gives $y = 1$ and the point $P_1 = (1, 1, 1)$ on $\mathcal{R}$.
      * Putting $x = 2$, gives $y = 3$ and the point $P_2 = (2, 3, 1)$ on $\mathcal{R}$.
      * Putting $x = 3$, gives $y = 5$ and the point $P_3 = (3, 5, 1)$ on $\mathcal{R}$.
      * Putting $x = 4$, gives $y = 0$ and the point $P_4 = (4, 0, 1)$ on $\mathcal{R}$.
      * Putting $x = 5$, gives $y = 2$ and the point $P_5 = (5, 2, 1)$ on $\mathcal{R}$.
      * Putting $x = 6$, gives $y = 4$ and the point $P_6 = (6, 4, 1)$ on $\mathcal{R}$.

We have $A = (1, 3, 6) = (-1, -3, -6) = (6, 4, 1) = P_6$ and $B = (3, 1, 5) = (9, 3, 15) = (2, 3, 1) = P_2$.

So the other points on $\mathcal{R}$ are $P_\infty, P_0, P_1, P_3, P_4$ and $P_5$. 
• Find the point of intersection of the line $\mathcal{R}$ with the line $\mathcal{S}$ with equation $x + 2y + 3z = 0$.

At the point of intersection, both equations hold at once, so we need:

$$x + 2y + 3z = 0, \quad -2x + y + z = 0.$$ 

Doubling the first equation and adding it to the second eliminates $x$, giving the equation:

$$0 = 2(x + 2y + 3z) - 2x + y + z = 2x + 4y + 6z - 2x + y + z = 5y.$$ 

So, after multiplying both sides by $\frac{1}{5} = 3$, we get $y = 0$.

Back substituting the condition $y = 0$ into the equation $-2x + y + z = 0$ gives the relation $z = 2x$, giving the point of intersection as $(x, 0, 2x)$ (with $x \neq 0$), or, after scaling, $(1, 0, 2)$.

It is easily seen that the point $(1, 0, 2)$ obeys both equations, so it gives the required point of intersection and we are done.

**Question 2**

Consider the $\mathbb{Z}_{13}$ projective geometry.

• Find the equation of the line $\mathcal{T}$ through the points $C = (1, 4, 1)$ and $D = (2, 3, 5)$.

What is the point at infinity of this line?

Using the determinant approach, the line has equation:

$$0 = \det \begin{vmatrix} x & y & z \\ 1 & 4 & 1 \\ 2 & 3 & 5 \end{vmatrix} = x \det \begin{vmatrix} 4 & 1 \\ 3 & 5 \end{vmatrix} - y \det \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} + z \det \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$$

$$= x(20 - 3) - y(5 - 2) + z(3 - 8) = 17x - 3y - 5z = 4x - 3y - 5z.$$ 

It is easy to check that both the points $C$ and $D$ satisfy this equation, so this is the correct equation for the line $\mathcal{T}$.

The point at infinity of $\mathcal{T}$ is obtained by putting $z = 0$, which gives $4x - 3y = 0$, with solution $(x, y, z) = s(3, 4, 0)$ for any $s \neq 0$.

So the point at infinity on $\mathcal{T}$ is $(3, 4, 0)$. 
• Find the condition on the number z that the point \((3, 5, z)\) lie on the line \(T\).

The point \((3, 5, z)\) lies on the line iff it obeys the equation of the line \(T\): \(4x - 3y - 5z = 0\), iff:

\[
0 = 4(3) - 3(5) - 5z = 12 - 15 - 5z = -3 - 5z = 10 - 5z, \quad z = 2.
\]

So the required point is \((3, 5, 2)\).

• Find the point \(p\) of intersection of the line \(T\) with the line \(U\) with equation \(2x + 3y + 2z = 0\).

What is the equation of the point \(p\) and what is the geometrical meaning of that equation?

Using the determinant approach, the point \(p\) has the equation:

\[
0 = \det \begin{vmatrix} a & b & c \\ 4 & -3 & -5 \\ 2 & 3 & 2 \end{vmatrix} = a \det \begin{vmatrix} -3 & -5 \\ 3 & 2 \end{vmatrix} - b \det \begin{vmatrix} 4 & -5 \\ 2 & 2 \end{vmatrix} + c \det \begin{vmatrix} 4 & -3 \\ 2 & 3 \end{vmatrix}
\]

\[
= a(-6 - (-15)) - b(8 - (-10)) + c(12 - (-6)) = 9a - 18b + 18c = 9(a - 2b + 2c).
\]

So we may take the equation of the point \(p\) to be \(a - 2b + 2c = 0\). The point \(p\) itself is then the point \((1, -2, 2)\), which can be seen to satisfy both the equations of the given lines, \(T\) and \(U\), so is their (necessarily unique) meeting point, as required.

Finally, the equation of the point \(p\), the equation \(a - 2b + 2c = 0\), gives the condition on the line co-ordinates \((a, b, c)\) of a line, that the line pass through the given point \(p\).