Topics in Geometry 05-3, Exam 1 Solutions, 7/14/5

Question 1

Show that the following list of axioms for a geometry are consistent.
Are the axioms categorical?

- **E1** There are exactly five points.
- **E2** There are exactly five lines.
- **E3** Each line goes through exactly three points.
- **E4** Every pair of points lies on a line.

In this geometry is it a theorem that every point lies on exactly three lines?
Is any axiom dependent on the other four axioms?
Explain your answers.

- **Model I**

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The first three axioms obviously hold.
The last axiom is verified by inspection: for each pair of points we list a line that it lies on:

- \( \{A, B : q\} \),
- \( \{A, C : r\} \),
- \( \{A, D : p\} \),
- \( \{A, E : p\} \),
- \( \{B, C : r\} \),
- \( \{B, D : s\} \),
- \( \{B, E : q\} \),
- \( \{C, D : s\} \),
- \( \{C, E : t\} \),
- \( \{D, E : t\} \).
The first three axioms obviously hold.
The last axiom is verified by inspection: for each pair of points we list
a line that it lies on:

- \{A, B : p\},
- \{A, C : p\},
- \{A, D : r\},
- \{A, E : s\},
- \{B, C : p\},
- \{B, D : r\},
- \{B, E : s\},
- \{C, D : t\},
- \{C, E : t\},
- \{D, E : t\}.

In Model II, point A lies on four lines: \(\{p, q, r, s\}\), so the suggested
theorem that every point lies on exactly three lines is false.
On the other hand in Model I every point lies on exactly three lines,
by inspection, since the sum of the entries in any row of the incidence
matrix is 3, so Model I is distinct from Model II and therefore the
axiom system is not categorical.

- If we negate axiom \(E_1\), we obtain, for example, the following geometry:

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This has exactly three points, exactly five lines and every line goes
through every point.
The axioms \(E_2, E_3, E_4\) and the negation of axiom \(E_1\) clearly hold.
• If we negate axiom \( E_2 \), we obtain, for example, the following geometry:

\[
\begin{array}{cccccc}
 & p & q & r & s & t \\
A & 1 & 1 & 1 & 0 & 0 \\
B & 0 & 1 & 1 & 1 & 0 \\
C & 0 & 0 & 1 & 1 & 1 \\
D & 1 & 0 & 0 & 1 & 1 \\
E & 1 & 1 & 0 & 0 & 1 \\
\end{array}
\]

This has exactly five points, exactly six lines and every line goes through exactly three points.

This is the same as our original geometry, Model I above, but with an extra three-point line, \( u \), so the axioms \( E_1, E_3, E_4 \) and the negation of axiom \( E_2 \) clearly hold.

• If we negate axiom \( E_3 \), we obtain, for example, the following geometry:

\[
\begin{array}{cccc}
 & p & q & r \\
A & 1 & 1 & 1 \\
B & 1 & 1 & 1 \\
C & 1 & 1 & 1 \\
D & 1 & 1 & 1 \\
E & 1 & 1 & 1 \\
\end{array}
\]

This has exactly five points, exactly five lines and every line goes through every point.

The axioms \( E_1, E_2, E_4 \) and the negation of axiom \( E_3 \) clearly hold.

• If we negate axiom \( E_4 \), we obtain, for example, the following geometry:

\[
\begin{array}{cccc}
 & p & q & r \\
A & 1 & 1 & 1 \\
B & 1 & 1 & 1 \\
C & 1 & 1 & 1 \\
D & 0 & 0 & 0 \\
E & 0 & 0 & 0 \\
\end{array}
\]

This has exactly five points, exactly five lines and each line goes through only the points \( A, B \) and \( C \).

The axioms \( E_1, E_2, E_3 \) and the negation of axiom \( E_4 \) clearly hold.

So each of the four axioms is independent of the other three.
Question 2

Consider the \( \mathbb{Z}_{11} \) affine geometry.

- Find the equation of the line \( AB \) through the following points:

\[ A = (3, 5) \text{ and } B = (7, 2). \]

The line \( AB \) has slope \( \frac{5-2}{3-7} = -\frac{3}{4} = \frac{5}{4} = 2. \)

So the equation of \( AB \) is: \( y-5 = 2(x-3), \) or \( y = 2x-1, \) or \( -2x+y+1 = 0, \) so it has line co-ordinates \([-2, 1, 1]\).

- Give a parametrization of the line \( AB \).

\[ x = t, y = 2t - 1. \]

Here \( t \) is arbitrary.

- Find the other points of the line \( AB \).

Varying \( t \) we get the other points of \( AB \) as:

\((0, 10), (1, 1), (2, 3), (4, 7), (5, 9), (6, 0), (8, 4), (9, 6), (10, 8)\).

- Find the intersection point \( C \) of the line \( AB \) with the line with equation \( 3x - 5y = 2. \) We need:

\[ 3t - 5(2t - 1) = 2, \]
\[ -7t + 5 = 2, \]
\[ t = \frac{3}{7} = \frac{14}{7} = 2. \]

So the intersection point is the point \((2, 3)\).

- Find a parametrization for all lines that pass through the point \( C \).

The lines with line co-ordinates \([-2, 1, 1]\) and \([3, -5, -2]\) both pass through \( C \), so the required parametrization can be taken to be:

\[ r[-2, 1, 1] + s[3, -5, -2] = [-2r + 3s, r - 5s, r - 2s]. \]

Here \( r \) and \( s \) are not both zero, but are otherwise arbitrary.
Question 3

Consider the $\mathbb{Z}_5$ projective geometry.

- Find the equation of the line $\mathcal{L}$ through the following points:
  
  $C = (2, 0, 1)$ and $D = (2, 3, 2)$.

  By the determinant method, the required line has the equation:

  
  \[
  \det \begin{pmatrix}
  x & y & z \\
  2 & 0 & 1 \\
  2 & 3 & 2 \\
  \end{pmatrix} = x \det \begin{pmatrix}
  0 & 1 \\
  3 & 2 \\
  \end{pmatrix} - y \det \begin{pmatrix}
  2 & 1 \\
  2 & 2 \\
  \end{pmatrix} + z \det \begin{pmatrix}
  2 & 0 \\
  2 & 3 \\
  \end{pmatrix}
  \]

  
  $= x(0 - 3) - y(4 - 2) + z(6 - 0) = 2x + 3y + z$.

  So the required line has line co-ordinates $[2, 3, 1]$.

- Determine the other points of the line $\mathcal{L}$.

  Putting $z = 0$ gives $2x + 3y = 0$, with solution $(x, y, 0) = (-3, 2, 0) = (2, 2, 0) = 2(1, 1, 0)$, so the point at infinity of the line is $(1, 1, 0)$.

  For the remaining points, we may take $z = 1$ and then $2x + 3y + 1 = 0$, or $2x - 2y + 6 = 0$, or $x - y + 3 = 0$, or $y = x + 3$.

  As $x$ varies, we get the points $(0, 3, 1), (1, 4, 1), (2, 0, 1) = C, (3, 1, 1)$ and $(4, 2, 1)$.

  We have $(2, 3, 2) = (6, 9, 6) = (1, 4, 1) = D$ so the other points of the line $CD$ are:

  $(1, 1, 0), (0, 3, 1), (3, 1, 1)$ and $(4, 2, 1)$.

- Find the point $p$ of intersection of the line $\mathcal{L}$ with the line $\mathcal{M}$ with equation $x + 2y + 2z = 0$.

  By the determinant method, the required point has the equation:

  
  \[
  \det \begin{pmatrix}
  p & q & r \\
  2 & 3 & 1 \\
  1 & 2 & 2 \\
  \end{pmatrix} = p \det \begin{pmatrix}
  3 & 1 \\
  2 & 2 \\
  \end{pmatrix} - q \det \begin{pmatrix}
  2 & 1 \\
  1 & 2 \\
  \end{pmatrix} + r \det \begin{pmatrix}
  2 & 3 \\
  1 & 2 \\
  \end{pmatrix}
  \]

  
  $= p(6 - 2) - q(4 - 1) + r(4 - 3) = 4p + 2q + r$.

  So the required intersection point is $(4, 2, 1)$. 
\begin{itemize}
  \item Determine the homogeneous line co-ordinates of the point \( p \) with respect to the base points \( C \) and \( D \).

We need to solve the equation:

\[(4, 2, 1) = s(2, 0, 1) + t(2, 3, 2) = (2s + 2t, 3t, s + 2t),\]

\[4 = 2s + 2t, \quad 2 = 3t, \quad 1 = s + 2t,\]

\[t = \frac{2}{3} = \frac{12}{3} = 4,\]

\[s = 1 - 2t = 1 - 8 = -7 = 3.\]

Check:

\[3(2, 0, 1) + 4(2, 3, 2) = (6, 0, 3) + (8, 12, 8) = (14, 12, 11) = (4, 2, 1).\]

So the required line co-ordinates are \((3, 4)\).\]
Question 4

Let the vertices of a square in the plane be labelled (in order around the square) by the integers mod 4.

- Interpret the transformation \( n \to n + 1 \mod 4 \) as a symmetry operation \( \mathcal{R} \) of the square, with a picture and give the name of the transformation.

  We have \( 0 \to 1, 1 \to 2, 2 \to 3 \) and \( 3 \to 0 \).
  This is a rotation through \( -\frac{\pi}{2} \) radians.

- Interpret the transformation \( n \to -n \mod 4 \) as a symmetry operation \( \mathcal{S} \) of the square, with a picture and give the name of the transformation.

  We have \( 0 \to 0, 1 \to 3, 2 \to 2 \) and \( 3 \to 1 \).
  This is a reflection through the leading \( 0-2 \) diagonal.

- Give formulas and pictures for the transformations \( \mathcal{R}^2, \mathcal{S}^2, \mathcal{R}\mathcal{S} \) and \( \mathcal{S}\mathcal{R} \).

  - \( \mathcal{R}^2 \) is a rotation through \( -\pi \), so maps: \( 0 \to 2, 1 \to 3, 3 \to 1 \) and \( 2 \to 0 \).
    This transformation can be written \( n \to n + 2 \mod 4 \).
  - \( \mathcal{S}^2 \) is the identity so maps: \( 0 \to 0, 1 \to 1, 1 \to 1 \) and \( 0 \to 0 \).
    This transformation can be written \( n \to n \mod 4 \).
  - \( \mathcal{R}\mathcal{S} \).
    First we reflect in the leading \( 0-2 \) diagonal and then we rotate through \( -\frac{\pi}{2} \).
    The net effect is a reflection in the line connecting the bisectors of the \( 0-3 \) edge and of the \( 1-2 \) edge.
    The transformation maps: \( 0 \to 3, 1 \to 2, 2 \to 1 \) and \( 3 \to 0 \). This transformation can be written \( n \to 3 - n \mod 4 \).
  - \( \mathcal{S}\mathcal{R} \).
    First we rotate through \( -\frac{\pi}{2} \) and then we reflect in the leading \( 0-2 \) diagonal.
    The net effect is a reflection in the line connecting the bisectors of the \( 0-1 \) edge and of the \( 2-3 \) edge.
    The transformation maps: \( 0 \to 1, 1 \to 0, 2 \to 3 \) and \( 3 \to 2 \).
    This transformation can be written \( n \to 1 - n \mod 4 \).
Question 5

Describe the symmetries of each of the following frieze patterns $\mathcal{X}$ and $\mathcal{Y}$ (understood to go on forever, forwards and backwards):

- **Pattern $\mathcal{X}$:** $\cdots <<>><<>><<>><<>><<>>\cdots$
  - We can translate through $4n$ spacing units where $n$ is any integer.
  - There is a centerline-symmetry.
  - We have vertical reflections, with the mirror passing through the center of any $<<>>$ or through the center of any $>><<$.
  - We have point symmetry about the center of any $<<>>$, or about the center of any $>><<$.
  - We have glide reflections: first a center-line reflection, followed by any translation through $4m$ units, where $m$ is any non-zero integer.
  - The symmetry type is: $\mathcal{F}_5$.

- **Pattern $\mathcal{Y}$:** $\cdots \land \land \land \land \land \land \land \land \land \cdots$
  - We can translate through $2n$ spacing units where $n$ is any integer.
  - There is no centerline-symmetry.
  - We have vertical reflections, with the mirror passing through the apex of any $\land$ or through the apex of any $\land$.
  - There is no point symmetry.
  - There are no glide reflections.
  - The symmetry type is: $\mathcal{F}_4$. 
Question 6


- Sketch the triangles $ABC$ and $PQR$.

- Prove that the triangles $ABC$ and $PQR$ are congruent.

We have

- $AB = B - A = [4, 4] - [2, 8] = [2, -4]$, $|AB| = \sqrt{2^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}$,
- $BC = C - B = [10, 7] - [4, 4] = [6, 3]$, $|BC| = \sqrt{6^2 + 3^2} = \sqrt{45} = 3\sqrt{5}$,
- $CA = A - C = [2, 8] - [10, 7] = [-8, 1]$, $|CA| = \sqrt{(-8)^2 + 1^2} = \sqrt{65}$,
- $PQ = Q - P = [11, -4] - [7, -2] = [4, -2]$, $|PQ| = \sqrt{4^2 + (-2)^2} = \sqrt{20} = 2\sqrt{5}$,
- $QR = R - Q = [14, 2] - [11, -4] = [3, 6]$, $|QR| = \sqrt{3^2 + 6^2} = \sqrt{45} = 3\sqrt{5}$,

So $|AB| = |PQ|$, $|BC| = |QR|$ and $|CA| = |RQ|$, so the triangles $ABC$ and $PQR$ are congruent by side-side-side.

Note also that $|AB|^2 + |BC|^2 = 20 + 45 = 65 = |CA|^2$, so by Pythagoras’ Theorem, the triangle $ABC$ is right-angled at $B$.

Also $|PQ|^2 + |QR|^2 = 20 + 45 = 65 = |RQ|^2$, so by Pythagoras’ Theorem, the triangle $PQR$ is right-angled at $Q$. 
Describe a Euclidean transformation that maps $ABC$ to $PQR$.

Is the transformation direct or indirect? Explain your answer.

First we translate the triangle so that $B$ goes to $Q$.

This is a translation by the vector $Q - B = [11, -4] - [4, 4] = [7, -8]$.

Then we just need to rotate counter-clockwise around $Q$ until the triangles coincide.

The slope of $BC$ is $\frac{3}{6} = \frac{1}{2}$ and of $QR$ is $\frac{6}{3} = 2$.

Let $\tan(\alpha) = \frac{1}{2}$, with $\alpha$ acute; then $\sin(\alpha) = \frac{\sqrt{5}}{5}$ and $\cos(\alpha) = \frac{2}{\sqrt{5}}$.

Then $\tan\left(\frac{\pi}{2} - \alpha\right) = 2$, so $BC$ is inclined at $\alpha$ to the vertical and $CA$ at $\frac{\pi}{2} - \alpha$ to the vertical, so the angle of rotation is $\theta = \frac{\pi}{2} - 2\alpha$.

Then we have $\sin(\theta) = \cos(2\alpha) = 2\cos^2(\alpha) - 1 = 2\left(\frac{4}{5}\right) - 1 = \frac{3}{5}$.

Also $\cos(\theta) = \sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) = \frac{4}{5}$.

So we need to rotate counter-clockwise around $Q$ by the angle:

$\theta = \arcsin\left(\frac{3}{5}\right) = 0.6435011088$ radians, or $36.86989764$ degrees.

The composition of these two transformations is a rotation, so is a direct transformation.

The matrix $R$ for this rotation is of the form:

$$R = \begin{bmatrix}
\frac{4}{5} & -\frac{3}{5} & s \\
\frac{3}{5} & \frac{4}{5} & t \\
0 & 0 & 1
\end{bmatrix}$$

Here $s$ and $t$ are to be found.

Acting on $B = [4, 4, 1]$ we need to get $Q = [11, -4, 1]$, which gives:

$$\begin{vmatrix}
11 \\
-4 \\
1
\end{vmatrix} = \begin{vmatrix}
\frac{4}{5} & -\frac{3}{5} & s \\
\frac{3}{5} & \frac{4}{5} & t \\
0 & 0 & 1
\end{vmatrix} \begin{vmatrix}
4 \\
4 \\
1
\end{vmatrix} = \begin{vmatrix}
\frac{s + \frac{4}{5}}{1} \\
\frac{t + \frac{28}{5}}{1} \\
1
\end{vmatrix}$$

So $s = 11 - 4 = \frac{51}{5}$ and $t = -4 - \frac{28}{5} = -\frac{48}{5}$.

So the desired rotation matrix $R$ is now:

$$R = \begin{bmatrix}
\frac{4}{5} & -3 & 51 \\
3 & 4 & -48 \\
0 & 0 & 5
\end{bmatrix}$$

Check:

$$R \cdot ABC = \begin{bmatrix}
\frac{4}{5} & -3 & 51 \\
3 & 4 & -48 \\
0 & 0 & 5
\end{bmatrix} \begin{bmatrix}
2 & 4 & 10 \\
8 & 4 & 7 \\
1 & 1 & 1
\end{bmatrix} = \begin{bmatrix}
7 & 11 & 14 \\
2 & -4 & 2 \\
1 & 1 & 1
\end{bmatrix} = PQR.$$