Topics in Geometry, Summer 2005
Homework Assignments
Homework 1, due for discussion Tuesday June 21st and to be turned in Wednesday June 22nd

- Find three proofs of Pythagoras Theorem. For at least one of these, analyze the underlying assumptions in the proof and be prepared to discuss these in class on Tuesday.

- For the following three questions, we first reprise the four point geometry axioms:
  - 4P1 There are exactly four distinct points.
  - 4P2 Through any two distinct points, there is exactly one line.
  - 4P3 Exactly two distinct points lie on any line.

- For the four point geometry, we observed that our model had the property that in it there were exactly three lines through every point. Decide with proof whether or not this result is actually a theorem derivable from the axioms.

- We also observed that our model had exactly six lines. Decide with proof whether or not this result is a theorem derivable from the axioms.

- Write out the proof of the independence of each of the axioms 4P1, 4P2 and 4P3 of the other two axioms, giving, for each part, appropriate pictures and incidence matrices.
Homework 2 due Tuesday June 28th

- Let $ABC$ be a triangle with $AB = AC$.
  Prove that the angles $ABC$ and $ACB$ are equal.
  Conversely, prove that, if in a triangle $ABC$, the angles $ABC$ and $ACB$ are equal, then $AB = AC$.

- Give a geometrical construction for the perpendicular bisector of a given line segment.
  Prove that your construction does indeed give the required bisector.

- Prove that the four-point geometry obeys the axioms A1-A4 for an affine plane:
  - A1: Four distinct points exist, no three collinear.
  - A2: There exists at least one line with exactly $n > 1$ points on it.
  - A3: Given two points, there is a unique line through the points.
  - A4: Given a line $m$ and a point $P$ not on $m$, there is exactly one line through $P$ parallel to $m$.

Can you also prove that this is the only affine geometry for the case $n = 2$?
In the case $n = 2$, how many models of the three axioms, (A2, A3, A4), taken together, are there?
Homework 3 due Tuesday July 5th

- Sketch the dual of the nine point affine plane (hint: use the thirteen point projective geometry).

- Consider the following axioms for a geometry:
  - **G1** There are exactly four points.
  - **G2** There are exactly four lines.
  - **G3** Any two points lie on a line.
  - **G4** Any two lines meet.

  Construct a model for this geometry, giving its incidence matrix. Discuss with proof whether or not each axiom is independent of the other three axioms.

- Consider the following axiom system for a geometry:
  - There are exactly three points.
  - Each line is on exactly two points.
  - There are exactly four lines.

  Describe with proof all the possible models. Give axioms that distinguish between the various models. Illustrate with pictures and for each picture give the associated incidence matrix. Also, for each model sketch its dual model.

- In the nine point affine geometry find the equation of the line \( \mathcal{L} \) through the points \((1, 0)\) and \((2, 1)\) and find the third point of the line \( \mathcal{L} \). Also find the equation of the line \( \mathcal{M} \) through \((1, 2)\) parallel to \( \mathcal{L} \). Also show that there is exactly one line parallel to both \( \mathcal{L} \) and to \( \mathcal{M} \) and give its equation.
Homework 4 due Thursday July 7th

- Let $A$ and $B$ be distinct points in the Euclidean plane.
  Give a geometrical construction for the circle $O$ on the diameter $AB$.
  Let $C$ be a point of the circle $O$ distinct from $A$ and $B$.
  Prove that the triangle $ABC$ is a right-angled triangle.
  Also give a geometrical construction of the tangent line at $C$.
  Include a proof that the tangent line at $C$ intersects the circle only at $C$.

- Find a parametrization for all lines in the $\mathbb{Z}_{11}$ projective geometry that pass through the point $(2, 4, 7)$.

- Find a parametrization for all the points on the $\mathbb{Z}_{11}$ projective line that passes through the point $(2, 4, 7)$ and the point $(1, 2, 3)$.
  Also find the other points of this line.

- Prove that there is a unique circle through any given three non-collinear points in the Euclidean plane and give a geometrical construction for the circle, with proof that the construction works.

- Show that the three points $P = (2, 3, -2)$, $Q = (1, 2, -4)$ and $R = (0, 1, -6)$ are collinear in the $\mathbb{Z}_{11}$ projective geometry and find the homogeneous parameters of each of the points with respect to the other two points.
Homework 5 due Tuesday July 12th

- Write the multiplication table of the dihedral group of order eight, the
symmetry group of the square.
(This is question 13, page 113 of Cederburg).

- Discuss the relation between the symmetry groups of a rhombus, a
rectangle and a square.

- Analyze the symmetries of the infinite string formed from repeating
the letter V: ...VVVVVVVVV...
Discuss the two cases, one where there is a gap between the adjacent
letters and the other where there is no gap.

- Let $R$ be the reflection in the $y$-axis, $S$ the reflection in the line $y = x$
and $T$ the reflection in the line $x + y = 1$.
Discuss the group generated by these transformations of the plane.
Homework 6 due Thursday July 14th

- Classify the symmetries of the frieze patterns on page 170 of Cederburg.
- Classify the symmetries of the wallpaper patterns on page 172 of Cederburg.
- Let $R$ be a rotation through $\frac{\pi}{4}$ about the origin in the plane. Let $S$ be a translation through $(2, 2)$ in the plane.

Give matrix representations for $R$, $S$, $R^2$, $RS$, $SR$ and $S^2$ and interpret each of these transformations geometrically.
Homework 7 due Tuesday July 19th

- Describe the Euclidean symmetries of the standard co-ordinate grid: the collection of all lines $x = m$ and all lines $y = n$, where $m$ and $n$ are integers.

- Give matrix formulas for the reflection $\mathcal{R}$ in the line $y = 2x - 1$ and the glide reflection $\mathcal{S}$, with axis the line $y = 2x - 1$, which maps the point $(3, 4)$ to $(7, 14)$. Describe the group that the symmetries $\mathcal{R}$ and $\mathcal{S}$ generate.

- Let $A = [2, 1], B = [4, -3], C = [3, 4], P = [6, 2], Q = [4, -2], R = [9, 3]$.
  - Prove that the triangles $ABC$ and $PQR$ are congruent.
  - Find the matrix representing the Euclidean transformation $\mathcal{T}$ that maps $ABC$ to $PQR$.
  - Identify the nature of the transformation $\mathcal{T}$, giving in particular its invariant points and lines, if any.
  - Does some positive integer power of $\mathcal{T}$ give the identity transformation? Explain your answer.
Homework 8 due Thursday July 21st

- Describe the Euclidean symmetries of the wallpaper pattern formed by tiling the plane with congruent regular hexagons, where there are three hexagons at each vertex and adjacent hexagons share a common edge. If each such hexagon is divided into six equilateral triangles by joining its vertices by straight line segments to its center, does the symmetry group change and if so, how? Explain your answer.

- Give matrix formulas for the rotation $R$ about the center $(4, 3)$, which maps the point $(1, -1)$ to the point $(7, -1)$. What is the angle of this rotation? Also find the image under the rotation $R$ of the line with equation $2x - 3y + 1 = 0$.

  - Prove that the triangles $ABC$ and $PQR$ are similar.
  - Find the matrix representing the affine transformation $T$ that maps $ABC$ to $PQR$.
  - Identify the nature of the transformation $T$, giving in particular its invariant points and lines, if any.
  - Does some positive integer power of $T$ give the identity transformation? Explain your answer.