#16 \[\int x \arccos x \, dx = \frac{x^2}{2} \arccos x - \left( \frac{x^2}{2} \cdot \frac{1}{\sqrt{1-x^2}} \right) \, dx\]

\[u = \arccos x \Rightarrow du = -\frac{1}{\sqrt{1-x^2}} \, dx\]

\[dv = x \, dx \Rightarrow v = \int x \, dx = \frac{x^2}{2}\]

\[= \frac{x^2}{2} \arccos x + \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} \, dx\]

By formula (10), with \(a = 1\)

\[= \frac{x^2}{2} \arccos x + \frac{1}{2} \left[ -\frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \arcsin x \right] + C\]

\[= \frac{x^2}{2} \arccos x - \frac{1}{4} \sqrt{1-x^2} + \frac{1}{4} \arcsin x + C\]

#22 \[y_1 = x \ln x, \quad y_2 = \sin x\]

\[A = \int (\sin x - x \ln x) \, dx\]

\[= \left[ \sin x \ln x - \int \sin x \, dx \right]_{0}^{1.75}\]

\[u = \ln x \Rightarrow du = \frac{1}{x} \, dx\]

\[dv = x \, dx \Rightarrow v = \frac{x^2}{2}\]

\[= \left[ \sin x \ln x - \left( \frac{x^2}{2} - \sin x \right) \right]_{0}^{1.75}\]
\[
A = (- \cos x) \left|^{1.75}_0 \right. - \left( \frac{x^2 \ln x}{2} \right. \left. \mid^{1.75}_0 - \frac{1}{2} \int_0^{1.75} \frac{x}{x} \, dx \right)
\]
\[
= \left[ - \cos (1.75) - (- \cos 0) \right] - \frac{1}{2} x^2 \ln x \left|^{1.75}_0 + \frac{1}{2} \int_0^{1.75} x \, dx \right.
\]
\[
= 1.18 - \frac{1}{2} \left( 1.75^2 \ln (1.75) - \lim_{x \to 0} x^2 \ln x \right) + \frac{1}{2} \frac{x^2}{2} \left|^{1.75}_0 \right.
\]

we need the limit, because \( \ln x \) is not defined at \( x = 0 \).

\[
= 1.18 - 1.857 + \frac{1}{2} \lim_{x \to 0} x^2 \ln x + 0.765 = 1.088 + \frac{1}{2} \lim_{x \to 0} x^2 \ln x
\]

\[
\lim_{x \to 0} x^2 \ln x = \lim_{x \to 0} \frac{\ln x}{\frac{1}{x^2}} = \lim_{x \to 0} \frac{\frac{1}{x}}{-2 \frac{x}{x^2}} = \frac{1}{2} \lim_{x \to 0} \left( -\frac{x^2}{2} \right) = 0
\]

\[
\Rightarrow A = 1.088 \text{ square units}
\]

\[
\int (k^2 - x^2)^n \, dx = \int (k - x)^n (k + x)^n \, dx
\]

\[
u = (k - x)^n \Rightarrow du = -n(k - x)^{n-1} \, dx
\]

\[
v = (k + x)^{n+1} \text{ (by substitution)}
\]

\[
= \frac{(k - x)^n (k + x)^{n+1}}{n+1} - \frac{n}{n+1} \int (k - x)^{n-1} (k + x)^{n+1} \, dx
\]

\[
= \frac{(k^2 - x^2)^n (k + x)}{n+1} + \frac{n}{n+1} \int (k^2 - x^2)^{n-1} (k + x)^2 \, dx
\]

\[
= \frac{(k^2 - x^2)^n k}{n+1} + \frac{x (k^2 - x^2)^n}{n+1} + \frac{n}{n+1} \int (k^2 - x^2)^{n-1} (k^2 + 2kx + x^2) \, dx
\]
\[
\frac{k(k^2-x^2)^n}{n+1} + \frac{x(k^2-x^2)^n}{n+1} + \frac{n}{n+1} \left[ k^2 \int (k^2-x^2)^{n-1} \, dx + 2k \int x(k^2-x^2)^{n-1} \, dx + \int x^2(k^2-x^2)^{n-1} \, dx \right]
\]

\[
A = \int x(k^2-x^2)^{n-1} \, dx = -\frac{1}{2} \int u^{n-1} \, du = -\frac{1}{2} \frac{u^n}{n} = -\frac{(k^2-x^2)^n}{2n}
\]

\[
u = k^2-x^2
\]
\[
\frac{du}{du} = 2x \, dx
\]
\[
\frac{du}{2} = x \, dx
\]

\[
B = \int x^2(k^2-x^2)^{n-1} \, dx = \int x \cdot x(k^2-x^2)^{n-1} \, dx
\]

\[
u = x = \int du = dx
\]
\[
\frac{dv}{dv} = x(k^2-x^2)^{n-1} \, dx = \int x(k^2-x^2)^{n-1} \, dx = -\frac{(k^2-x^2)^n}{2n}
\]

\[
\text{this is what we called } A
\]

\[
\text{Put everything back into the equation} (1)
\]

\[
\int (k^2-x^2)^n \, dx = \frac{k(k^2-x^2)^n}{n+1} + \frac{x(k^2-x^2)^n}{n+1} + \frac{n}{n+1} \int (k^2-x^2)^{n-1} \, dx
\]

\[
+ \frac{2k}{n+1} \cdot \left( -\frac{(k^2-x^2)^n}{2n} \right) + \frac{2x}{n+1} \cdot \left( -\frac{x(k^2-x^2)^n}{2n} \right) + \frac{n}{n+1} \cdot \frac{1}{2a} \int (k^2-x^2)^n \, dx
\]
Note that we have the same integral in both sides: move the one in the RHS to the LHS:

\[ I - \frac{1}{2(n+1)} I = \frac{k(k^2-x^2)^n}{n+1} + x\frac{(k^2-x^2)^n}{n+1} + \frac{n k^2}{n+1} \int (k^2-x^2)^{n-1} dx \]

\[ - \frac{k(k^2-x^2)^n}{n+1} - \frac{x(k^2-x^2)^n}{2(n+1)} \]

\[ \frac{2n+2-1}{2(n+1)} I = x(k^2-x^2) + \frac{n k^2}{n+1} \int (k^2-x^2)^{n-1} dx \]

\[ \Rightarrow (2n+1)I = x(k^2-x^2) + 2n k^2 \int (k^2-x^2)^{n-1} dx \]

\[ \Rightarrow I = \frac{x(k^2-x^2)}{2n+1} + \frac{2n k^2}{2n+1} \int (k^2-x^2)^{n-1} dx \]

There is another way of solving this, but it requires a "little bit" of creativity: use integration by parts this way:

\[ u = (k^2-x^2)^n \Rightarrow du = n(k^2-x^2)^{n-1} (-2x) dx \]

\[ dv = dx \Rightarrow v = x \]

\[ \int (k^2-x^2)^n dx = x(k^2-x^2)^n - 2n \int -x^2(k^2-x^2)^{n-1} dx \]

Now, add and subtract \(2n \int k^2(k^2-x^2)^{n-1} dx\) in the right hand side:

\[ = x(k^2-x^2)^n - 2n \int -x^2(k^2-x^2)^{n-1} dx \]

\[ - 2n \int k^2(k^2-x^2)^{n-1} dx + 2n \int k^2(k^2-x^2)^{n-1} dx \]
Combine the second and third integral:

\[ \int (k^2-x^2)^n \, dx = x(k^2-x^2) - 2n \int (k^2-x^2) \, dx + 2nk^2 \int (k^2-x^2)^{n-1} \, dx \]

\[ n = 3, k = 1 \]

\[ k = 1 \]

\[ \int (1-x^3)^5 \, dx = x(1-x^3)^4 \bigg|_0^1 + \frac{10}{11} \int (1-x^2)^3 \, dx \]

\[ n = 3, k = 1 \]

\[ \frac{10}{11} \cdot \frac{8}{9} \cdot \frac{6}{5} \cdot \frac{4}{3} \cdot x^{-x^3} \bigg|_0^1 = \frac{10}{11} \cdot \frac{8}{9} \cdot \frac{6}{5} \cdot \frac{4}{3} \cdot \frac{2}{3} = \frac{3840}{10395} = \frac{256}{693} \]
\[
\frac{x^3 + 2x - 1}{(x-1)(2x+1)} = \frac{x^3 + 2x - 1}{2x^2 - x - 1} = \frac{1}{2} x + \frac{1}{4} + \frac{\frac{11}{4} x - \frac{3}{4}}{(x-1)(2x+1)}
\]

\[
2x^2 - x - 1
\]

\[
\frac{1}{2} x + \frac{1}{4}
\]

\[
\frac{x^3 + 2x^2 + 2x - 1}{2x^2 - x - 1}
\]

\[
\frac{1}{2} x^2 - \frac{1}{2} x - \frac{1}{4}
\]

\[
\frac{1}{2} x^2 + \frac{5}{2} x - 1
\]

\[
\frac{1}{2} x^2 - \frac{1}{2} x - \frac{1}{4}
\]

\[
\frac{11}{4} x - \frac{3}{4}
\]

\[
\begin{align*}
Q(x) &= \frac{1}{2} x + \frac{1}{4} \\
R(x) &= \frac{11}{4} x - \frac{3}{4}
\end{align*}
\]

We integrate now:

\[
\int \frac{x^3 + 2x - 1}{(x-1)(2x+1)} \, dx = \int \left( \frac{1}{2} x + \frac{1}{4} + \frac{\frac{11}{4} x - \frac{3}{4}}{(x-1)(2x+1)} \right) \, dx
\]

\[
= \frac{1}{2} x^2 + \frac{1}{4} x + \frac{1}{4} \int \frac{11x - 3}{(x-1)(2x+1)} \, dx
\]

\[
\frac{11x - 3}{(x-1)(2x+1)} = \frac{A}{x-1} + \frac{B}{2x+1}
\]

\[
11x - 3 = A(2x + 1) + B(x - 1)
\]

\[
x = -\frac{1}{2} \Rightarrow -\frac{11}{2} - 3 = 0 - \frac{3B}{2} \Rightarrow -\frac{17}{2} = -\frac{3B}{2} \Rightarrow B = \frac{17}{3}
\]

\[
x = 1 \Rightarrow 8 = 3A \Rightarrow A = \frac{8}{3}
\]
\[ \frac{11x - 3}{(x-1)(2x+1)} = \frac{8}{3} \frac{1}{x-1} + \frac{17}{3} \frac{1}{2x+1} \]

\[ = \int \frac{11x - 3}{(x-1)(2x+1)} \, dx = \frac{8}{3} \int \frac{1}{x-1} \, dx + \frac{17}{3} \int \frac{1}{2x+1} \, dx \]

\[ u = x-1 \quad \quad v = 2x+1 \]
\[ du = dx \quad \quad dv = 2 \, dx \quad \Rightarrow \frac{dv}{2} = dx \]

\[ = \frac{8}{3} \left( \int \frac{du}{u} + \frac{17}{3} \cdot \frac{1}{2} \int \frac{1}{v} \, dv \right) \]
\[ = \frac{8}{3} \ln |u| + \frac{17}{6} \ln |v| = \frac{8}{3} \ln |x-1| + \frac{17}{6} \ln |2x+1| \]

\[ = \int \frac{x^3 + 2x - 1}{(x-1)(2x+1)} \, dx = \int \left( \frac{x^2}{4} + \frac{x}{4} + \frac{1}{4} \left( \frac{8}{3} \ln |x-1| + \frac{17}{6} \ln |2x+1| \right) \right) \, dx \]
\[ = \frac{x^2}{4} + \frac{x}{4} + \frac{2}{3} \ln |x-1| + \frac{17}{24} \ln |2x+1| + C \]

\[ \frac{2x^3 - x^2 - x + 6}{x^2 - 1} = 2x - 1 + \frac{x + 5}{x^2 - 1} \]

\[ = \frac{2x - 1}{x^2 - 1} \]

\[ \frac{2x^3 - x^2 - x + 6}{x^2 - 1} \]

\[ \frac{2x^3 - x^2 - x + 6}{x^2 - 1} = \int \left( 2x - 1 + \frac{x + 5}{(x+1)(x-1)} \right) \, dx \]
\[ = 2 \int \frac{x^2}{2} - x + \int \frac{x + 5}{(x+1)(x-1)} \, dx \]
\[
\frac{x+5}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} = -\frac{2}{x+1} + \frac{3}{x-1}
\]

\[
x+5 = A(x-1) + B(x+1)
\]

\[
x = 1 \implies 6 = 2B \implies B = 3
\]

\[
x = -1 \implies 4 = -2A \implies A = -2
\]

\[
\Rightarrow \int \frac{2x^3-x^2-x+6}{x^2-1} \, dx = x^2 - x + \left( \frac{3}{x-1} - \frac{2}{x+1} \right) \, dx
\]

\[
= x^2 - x + 3 \ln |x-1| - 2 \ln |x+1| + C
\]

5.9 (20) \[ y' = \frac{t \cos t}{y^2} ; \quad y\left(\frac{\pi}{2}\right) = 3 \]

\[
\frac{dy}{dt} = \frac{t \cos t}{y^2} \quad \Rightarrow \quad y^2 \, dy = t \cos t \, dt
\]

\[
\int y^2 \, dy = \int t \cos t \, dt
\]

\[
u = t \implies dv = dt
\]

\[
du = \cos t \, dt \implies v = \sin t
\]

\[
\Rightarrow \frac{y^3}{3} = t \sin t - \int \sin t \, dt
\]

\[
\frac{y^3}{3} = t \sin t + \cos t + C
\]

\[
\frac{3^3}{3} = \frac{\pi}{2} \sin \left( \frac{\pi}{2} \right) + \cos \left( \frac{\pi}{2} \right) + C
\]

\[
\Rightarrow q = \frac{\pi}{2} + C \Rightarrow C = q - \frac{\pi}{2}
\]

\[
\Rightarrow y^3 = 3 \left( t \sin t + \cos t + q - \frac{\pi}{2} \right) \quad \Rightarrow \quad y = \sqrt[3]{3 \left( t \sin t + \cos t + q - \frac{\pi}{2} \right)}
\]