Integrated Calculus II 05-2, Quiz 5, 4/8/5

Question 1

Consider the following series: \( S(x) = \sum_{n=1}^{\infty} \frac{x^n}{n2^n} \).

- Implement the ratio test:
  
  \[ a_n = \frac{x^n}{n2^n} \]

  \[ a_{n+1} = \frac{x^{n+1}}{(n+1)2^{n+1}} \]

  \[ \frac{a_{n+1}}{a_n} = \left( \frac{x^{n+1}}{(n+1)2^{n+1}} \right) \left( \frac{n2^n}{x^n} \right) = \left( \frac{n}{n+1} \right) \left( \frac{x}{2} \right) \]

  \[ \rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \left( \frac{n}{n+1} \right) \left( \frac{x}{2} \right) \right| = \left| \frac{x}{2} \right|. \]

  By the ratio test the series converges if \( \rho < 1 \), so if \( \left| \frac{x}{2} \right| < 1 \), so if \( |x| < 2 \), so if \(-2 < x < 2\).

  Also by the ratio test says, the series diverges if \( \rho > 1 \), so if \( \left| \frac{x}{2} \right| > 1 \), so if \( |x| > 2 \), so if \( x > 2 \) or if \( x < -2 \). The ratio test by itself gives no information when \( \rho = 1 \), so when \( \left| \frac{x}{2} \right| = 1 \), so if \( |x| = 2 \), so if \( x = 2 \) or if \( x = -2 \).

  Using this limit the ratio test tells us that the series converges on the open interval: \( J = (-2, 2) \).

- Does the series converge at the right-end of the interval \( J \)?
  
  No.

  Explain your answer.

  We put \( x = 2 \) in the series \( S(x) \), giving the series:

  \[ \sum_{n=1}^{\infty} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \]

  This we recognize as the standard harmonic series, which diverges.
Does the series converge at the left-end of the interval \( J \)?
Yes.

Explain your answer.
We put \( x = -2 \) in the series \( S(x) \), giving the series:

\[
\sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots
\]

This we recognize as the negative of the standard alternating harmonic series, so it converges (with sum \(-\ln(2)\)), by the alternating series test:

- The series is alternating.
- The size of the \( n \)-the term \( a_n = \frac{(-1)^n}{n} \) is \( \frac{1}{n} \), which decreases as \( n \) increases.
- As \( n \to \infty \), \( a_n \to 0 \).

Putting \( x = -1 \), estimate \( S(-1) \) using the first five terms of the series.

We have to the fifth term:

\[
S(-1) = \sum_{n=1}^{5} \frac{(-1)^n}{n2^n} = -\frac{1}{2} + \frac{1}{2(2^2)} - \frac{1}{3(2^3)} + \frac{1}{4(2^4)} - \frac{1}{5(2^5)}
\]

\[
= -\frac{1}{2} + \frac{1}{8} - \frac{1}{24} + \frac{1}{64} - \frac{1}{160} = \frac{-480 + 120 - 40 + 15 - 6}{960} = \frac{-391}{960} = -0.4072916667.
\]

What kind of series is obtained?
The series is an alternating series and converges by the alternating series test.

Determine the maximum possible error in your estimate of \( S(-1) \).
The maximum error in the estimate is the next term in the series which is \( \frac{1}{6(2^6)} = \frac{1}{384} \), or about 0.0026.
Write out the first five terms of the series \( S(x) \) and by differentiating, give the first five terms of the series for \( S'(x) \).

What kind of series is obtained?

Sum the series for \( S'(x) \) exactly.

By integrating your answer, obtain a formula for \( S(x) \).

Hence evaluate \( S(-1) \) and compare with your numerical calculations.

\[
S(x) = \sum_{n=1}^{\infty} \frac{x^n}{n2^n} = \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{24} + \frac{x^4}{64} + \frac{x^5}{160} + \ldots
\]

\[
S'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n2^n} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{2^n} = \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \frac{x^4}{32} + \ldots
\]

This we recognize as geometric, with ratio \( r = \frac{x}{2} \) and first term \( a = \frac{1}{2} \), so its sum is:

\[
S'(x) = \frac{a}{1 - r} = \frac{\frac{1}{2}}{1 - \frac{x}{2}} = \frac{1}{2 - x}.
\]

This sum is valid provided \( |r| < 1 \), or \( |\frac{x}{2}| < 1 \), so provided \(-2 < x < 2\).

Integrating we get:

\[
S(x) = -\ln(2 - x) + C.
\]

Putting \( x = 0 \) in the series \( S(x) \) we get \( S(0) = 0 \), so we get:

\[
0 = -\ln(2) + C; \quad C = \ln(2),
\]

\[
S(x) = \ln(2) - \ln(2 - x) = \ln \left( \frac{2}{2 - x} \right).
\]

Putting \( x = -1 \), we get \( S(-1) = \ln \left( \frac{2}{3} \right) = -0.40547 \).

The actual error in our estimate is then about 0.00183 or about two-thirds of the maximum possible error.
Question 2

Obtain the series for \( \text{arctan}(x) \) to order \( x^{11} \) by the following method:

- The standard geometric series for \( \frac{1}{1-x} \) to order \( x^5 \) is:

\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \sum_{n=0}^\infty x^n.
\]

The full series is convergent if and only if \(-1 < x < 1\).

- Replacing \( x \) by \(-x\), we get the series for \( \frac{1}{1+x} \) to order \( x^5 \) as:

\[
\frac{1}{1+x} = 1 + (-x) + (-x)^2 + (-x)^3 + (-x)^4 + (-x)^5 + \cdots = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots = \sum_{n=0}^\infty (-1)^n x^n.
\]

The full series is convergent if and only if \(-1 < x < 1\).

- Replacing \( x \) by \( x^2 \), we get the series for \( \frac{1}{1+x^2} \) to order \( x^{10} \) as:

\[
\frac{1}{1+x^2} = 1 - x^2 + (x^2)^2 - (x^2)^3 + (x^2)^4 - (x^2)^5 + \cdots = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots = \sum_{n=0}^\infty (-1)^n x^{2n}.
\]

The full series is convergent if and only if \(-1 < x < 1\).

- Integrating term by term we get the series for \( \int \frac{1}{1+x^2} \) to order \( x^{11} \) as (explain why the integration constant is zero):

\[
\text{arctan}(x) + C = \int \frac{1}{1+x^2} \, dx = \int (1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots) \, dx
\]

\[
= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots.
\]
Putting \( x = 0 \) in both sides of this equation, we get \( \arctan(0) + C = 0 \), so \( C = 0 \):

\[
\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.
\]

The ratio test gives:

\[
\rho = \lim_{n \to \infty} \left| \frac{\frac{2n+1}{2n+3} \frac{(2n+1)(-1)^n x^{2n+3}}{(-1)^{n+1} x^{2n+3}}}{\frac{2n+3}{2n+1}} \right| = x^2 \lim_{n \to \infty} \left| \frac{2n+1}{2n+3} \right| = x^2.
\]

So by the ratio test, the full series converges for \(-1 < x < 1\) and is divergent for \(x > 1\) or \(x < -1\).

When \( x = -1 \), the series is the negative of the series when \( x = 1 \), so we need only analyze the series for the end-point \( x = 1 \).

When \( x = 1 \), the series is:

\[
1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.
\]

This series converges by the alternating series test:

- The series is alternating.
- The size of the \( n \)-the term \( a_n = \frac{(-1)^n}{2n+1} \) is \( \frac{1}{2n+1} \), which decreases as \( n \) increases.
- As \( n \to \infty \), \( a_n \to 0 \).

So the series converges for \(-1 \leq x \leq 1\) and diverges for all other values of \( x \).

By putting \( x = 1 \) in your series, estimate \( \pi = 4 \arctan(1) \) and give an error estimate.

It can be shown that at \( x = 1 \), the series sum is then \( \arctan(1) = \frac{\pi}{4} \).

Keeping only the first six terms of the series, we get the estimate for \( \frac{\pi}{4} \):

\[
1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} = \frac{3465 - 1155 + 693 - 495 + 385 - 315}{3465} = \frac{2578}{3465} = 0.744.
\]

Since the series is alternating, the maximum possible error is the size of the next term \( \frac{1}{13} = 0.077 \).

Multiplying by 4, we get a pretty rough estimate of \( \pi \) as \( \frac{10312}{315} = 2.976 \) with an error of no more that \( \frac{4}{315} = 0.128 \).
Question 3

Let \( f(x) = \tan(x) \), defined for \(-\frac{\pi}{2} < x < \frac{\pi}{2}\).

- Using Maple, obtain the first seven derivatives of \( f(x) = \tan(x) \) and evaluate these derivatives at the origin, \( x = 0 \).

\[
\begin{align*}
  f(x) &= \tan(x), \quad f(0) = 0, \\
  f'(x) &= \sec^2(x), \quad f'(0) = 1, \\
  f''(x) &= 2 \sec^2(x) \tan(x), \quad f''(0) = 0, \\
  f'''(x) &= 4 \sec^2(x) \tan^2(x) + 2 \sec^4(x) = 6 \sec^4(x) - 4 \sec^2(x), \quad f'''(0) = 2, \\
  f^{(4)}(x) &= (24 \sec^4(x) - 8 \sec^2(x)) \tan(x), \quad f^{(4)}(0) = 0, \\
  f^{(5)}(x) &= (24 \sec^4(x) - 8 \sec^2(x)) \sec^2(x) + (96 \sec^4(x) - 16 \sec^2(x)) \tan^2(x) \\
  &= 120 \sec^6(x) - 120 \sec^4(x) + 16 \sec^2(x), \quad f^{(5)}(0) = 16, \\
  f^{(6)}(x) &= (720 \sec^6(x) - 480 \sec^4(x) + 32 \sec^2(x)) \tan(x), \quad f^{(6)}(0) = 0, \\
  f^{(7)}(x) &= (720 \sec^6(x) - 480 \sec^4(x) + 32 \sec^2(x)) \sec^2(x) \\
  &\quad + (4320 \sec^6(x) - 1920 \sec^4(x) + 64 \sec^2(x)) \tan^2(x), \\
  &= 5040 \sec^8(x) - 6720 \sec^6(x) + 2016 \sec^4(x) - 64 \sec^2(x), \quad f^{(7)}(0) = 272, \\
  f^{(8)}(x) &= (40320 \sec^8(x) - 40320 \sec^6(x) + 8064 \sec^4(x) - 128 \sec^2(x)) \tan(x).
\end{align*}
\]

- Hence determine the Taylor series to order seven \( T_7 \) of the function \( \tan(x) \) based at the origin.

The Taylor series to order seven, based at the origin, is:

\[
T_7 = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \frac{f^{(5)}(0)x^5}{5!} + \frac{f^{(6)}(0)x^6}{6!} + \frac{f^{(7)}(0)x^7}{7!}
\]

\[
= x + \frac{2x^3}{6} + \frac{16x^5}{120} + \frac{272x^7}{5040}
\]

\[
= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315}.
\]
• Estimate \( \tan\left(\frac{1}{2}\right) \) and give an estimate of the error.

Putting \( x = \frac{1}{2} \) in the series gives the estimate of \( \tan\left(\frac{1}{2}\right) \) as:

\[
\frac{1}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{2\left(\frac{1}{2}\right)^5}{15} + \frac{17\left(\frac{1}{2}\right)^7}{315} = \frac{1}{2} + \frac{1}{24} + \frac{1}{240} + \frac{17}{40320}
\]

\[
= \frac{1}{40320} (20160 + 1680 + 168 + 17) = \frac{22025}{40320} = \frac{4405}{8064} = 0.5462549603174.
\]

Plotting \(|f^{(8)}(x)|\) on the interval \([-\frac{1}{2}, \frac{1}{2}]\) we see that its maximum value \(K_8\) is attained at \( x = \frac{1}{2} \) and is about 21727.55656.
Then the Taylor series error formula gives the maximum possible error in our estimate as:

\[
\frac{1}{8!}K_8(\frac{1}{2})^8 = 0.002105.
\]

Maple gives \( \tan\left(\frac{1}{2}\right) = 0.546302 \), so the actual error is about 0.0000475295, well within our estimate.

• Multiply \( T_7 \) by the standard series for \( \cos(x) \) based at the origin, to order seven and show that the resulting series gives the standard series for \( \sin(x) \) to order seven.

The series for \( \cos(x) \) to order seven is:

\[
\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}.
\]

Multiplying by the series \( T_7 \) keeping only terms to \( x^7 \) gives:

\[
(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720})(x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315})
= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} - (\frac{x^3}{2} + \frac{x^5}{6} + \frac{x^7}{15}) + (\frac{x^5}{24} + \frac{x^7}{72} - \frac{x^7}{720})
= x - \frac{x^3}{6} + x^5(\frac{2}{15} - \frac{1}{6} + \frac{1}{24}) + x^7(\frac{17}{315} - \frac{1}{15} + \frac{1}{72} - \frac{1}{720})
= x - \frac{x^3}{6} + x^5(16 - 20 + 5) + \frac{x^7}{5040}(252 - 336 + 70 - 7)
= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}.
\]

To this order this exactly agrees with the series for \( \sin(x) \), as it should, since \( \cos(x) \tan(x) = \sin(x) \).
• Differentiate $T_7$ with respect to $x$, multiply twice by the series for $\cos(x)$ to order seven, keeping only terms to order $x^7$. Explain your answer.

$$T_7' = 1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45}.$$  

$$\cos(x)T_7' = (1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45})(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720})$$

$$= 1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} - \frac{x^2}{2} - \frac{x^4}{2} - \frac{x^6}{3} + \frac{x^4}{24} + \frac{x^6}{24} - \frac{x^6}{720}$$

$$= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720}$$

$$\cos^2(x)T_7' = (1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720})(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720})$$

$$= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} - \frac{x^2}{2} - \frac{x^4}{4} - \frac{5x^6}{48} + \frac{x^4}{24} + \frac{x^6}{48} - \frac{x^6}{720}$$

$$= 1.$$  

This result is to be expected, because we have the derivative:

$$\frac{d}{dx}\tan(x) = \sec^2(x) = \frac{1}{\cos^2(x)}.$$  

Multiplying both sides by $\cos^2(x)$ then gives the relation:

$$\cos^2(x)\frac{d}{dx}\tan(x) = 1.$$