Question 1

Evaluate each of the following integrals, showing your work.

- \( \int_{0}^{\frac{\pi}{2}} \sin(x) \cos^4(x) \, dx \).

Here we substitute:
- \( u = \cos(x) \).
- \( du = \frac{du}{dx} \, dx = -\sin(x) \, dx \).
- The differential becomes: \( \sin(x) \cos^4(x) \, dx = -u^4 \, du \).
- \( u = \cos(0) = 1 \) when \( x = 0 \) and \( u = \cos(\frac{\pi}{3}) = \frac{1}{2} \), when \( x = \frac{\pi}{3} \).

Then the integral becomes:

\[
\int_{0}^{\frac{\pi}{2}} \sin(x) \cos^4(x) \, dx = \int_{1}^{\frac{1}{2}} u^4 \, (-du) = - \left[ \frac{u^5}{5} \right]_{1}^{\frac{1}{2}} = - \frac{1}{160} + \frac{1}{5} = \frac{31}{160}.
\]

- \( \int_{0}^{\frac{\pi}{2}} x \cos(2x) \, dx \).

Here we integrate by parts:
- \( u = x \).
- \( dv = \cos(2x) \, dx \).
- \( du = \, dx \).
- \( v = \int dv = \int \cos(2x) \, dx = \frac{1}{2} \sin(2x) \).

Then the integral becomes:

\[
\int_{0}^{\frac{\pi}{2}} x \cos(2x) \, dx = \left[ \frac{x \, \sin(2x)}{2} \right]_{0}^{\frac{\pi}{2}} - \int_{0}^{\frac{\pi}{2}} \frac{1}{4} \, \cos(2x) \, dx = 0 - 0 + \left[ \frac{1}{4} \cos(2x) \right]_{0}^{\pi} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.
\]
\[ \int_{0}^{\frac{\pi}{4}} x \sec^2(x) \, dx. \]

We integrate by parts:

- \( u = x, \)
- \( dv = \sec^2(x) \, dx, \)
- \( du = dx, \)
- \( v = \int dv = \int \sec^2(x) \, dx = \tan(x). \)

Then the integral becomes:

\[
\int_{0}^{\frac{\pi}{4}} x \sec^2(x) \, dx = \left[ uv \right]_{x=0}^{x=\frac{\pi}{4}} - \int_{x=0}^{x=\frac{\pi}{4}} v \, du
\]

\[
= [x \tan(x)]_{0}^{\frac{\pi}{4}} - \int_{0}^{\frac{\pi}{4}} \tan(x) \, dx
= \frac{\pi}{4} \tan(\frac{\pi}{4}) + \ln(\cos(x))\bigr|_{0}^{\frac{\pi}{4}}
= \frac{\pi}{4} + \ln(\cos(\frac{\pi}{4})) - \ln(\cos(0)) = \frac{\pi}{4} + \ln(\frac{1}{2^{\frac{1}{2}}}) - \ln(1) = \frac{\pi}{4} - \frac{1}{2} \ln(2).
\]

Here we used the integral:

\[
\int \tan(x) \, dx = -\ln(A \cos(x)).
\]

This is a standard substitution:

- \( u = \cos(x), \)
- \( du = \frac{du}{dx} \, dx = -\sin(x) \, dx, \)

- The differential is then: \( \tan(x) \, dx = \frac{\sin(x) \, dx}{\cos(x)} = -\frac{du}{u}. \)

Then the integral is:

\[
\int \tan(x) \, dx = \int -\frac{du}{u} = -\ln(Au) = -\ln(A \cos(x)).
\]
\[ \int_{-3}^{4} \sqrt{25-t^2} \, dt. \]

Here we make the substitution:

- \( t = 5 \sin(x) \), so also \( x = \arcsin\left(\frac{t}{5}\right) \).
- \( dt = 5 \cos(x) \, dx \).

\[ \sqrt{25-t^2} = \sqrt{25-25 \sin^2(x)} = \sqrt{25 \cos^2(x)} = 5 \cos(x). \]

- The differential becomes: \( \sqrt{25-t^2} \, dt = 5 \cos(x) \sin(x) \, dx = 5 \cos^2(x) \, dx. \)
- When \( t = -3 \), we have \( x = -\arcsin\left(\frac{3}{5}\right) \) and when \( t = 4 \), we have \( x = \arcsin\left(\frac{4}{5}\right) \).

Put \( \alpha = \arcsin\left(\frac{3}{5}\right) \) and \( \beta = \arcsin\left(\frac{4}{5}\right) \).

Then \( \alpha \) and \( \beta \) are acute angles and we have:

\[ \sin(\alpha) = \frac{3}{5}, \quad \cos(\alpha) = \sqrt{1-\sin^2(\alpha)} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5} = \sin(\beta). \]

It follows that \( \alpha \) and \( \beta \) are complementary angles: \( \alpha + \beta = \frac{\pi}{2} \).

Then the integral becomes:

\[
\int_{-3}^{4} \sqrt{25-t^2} \, dt = \int_{-\alpha}^{\beta} 25 \cos^2(x) \, dx = \frac{25}{2} \int_{-\alpha}^{\beta} (1 + \cos(2x)) \, dx
\]

\[
= \frac{25}{2} \left[ x + \frac{1}{2} \sin(2x) \right]_{-\alpha}^{\beta}
= \frac{25}{2} \left[ x + \sin(x) \cos(x) \right]_{-\alpha}^{\beta}
= \frac{25}{2} \left( \beta + \sin(\beta) \cos(\beta) - (-\alpha + \sin(-\alpha) \cos(-\beta)) \right)
= \frac{25}{2} \left( \alpha + \beta + 2 \left( \frac{3}{5} \right) \left( \frac{4}{5} \right) \right)
= \frac{25}{2} \left( \frac{\pi}{2} + \frac{24}{25} \right)
= \frac{25\pi}{4} + 12.
\]

Here we also used the double angle formulas:

- \( \cos^2(x) = \frac{1}{2}(1 + \cos(2x)). \)
- \( 2 \sin(x) \cos(x) = \sin(2x). \)

Note that plotting the curve, \( y = \sqrt{25-t^2} \) for \(-3 \leq t \leq 4\), we see that the given integral gives the area of a quadrant of the circle of radius 5: \( t^2 + y^2 = 25 \) added to twice the area of a triangle of base 3 and height 4, so the correct answer may be obtained without any calculus!
\begin{itemize}
\item \( \int_1^e \frac{\ln(t)}{t} \, dt \).
\end{itemize}

We substitute:
\begin{itemize}
\item \( u = \ln(t) \),
\item \( du = \frac{dt}{t} \).
\end{itemize}
- When \( t = 1 \), we have \( u = \ln(1) = 0 \).
- When \( t = e \), we have \( u = \ln(e) = 1 \).

So we get:
\[ \int_1^e \frac{\ln(t)}{t} \, dt = \int_0^1 u \, du = \left[ \frac{1}{2} u^2 \right]_0^1 = \frac{1}{2} \]

\begin{itemize}
\item \( \int_1^{e^2} t \ln(t) \, dt \).
\end{itemize}

We integrate by parts:
\begin{itemize}
\item \( u = \ln(t) \),
\item \( dv = t \, dt \),
\item \( du = \frac{dt}{t} \),
\item \( v = \int dv = \frac{1}{2} t^2 \).
\end{itemize}

Then the indefinite integral becomes:
\[ \int t \ln(t) \, dt = \int u \, dv = uv - \int v \, du = \frac{1}{2} t^2 \ln(t) - \int \frac{1}{2} t^2 \frac{dt}{t} \]
\[ = \frac{1}{2} t^2 \ln(t) - \int \frac{1}{2} t \, dt = \frac{1}{2} t^2 \ln(t) - \frac{1}{4} t^2 = \frac{t}{4} (2t \ln(t) - t) + C. \]

The definite integral is then:
\[ \int_1^{e^2} t \ln(t) \, dt = \left[ \frac{t}{4} (2t \ln(t) - t) \right]_1^{e^2} \]
\[ = \frac{e^2}{4} (2e^2 \ln(e^2) - e^2) - \frac{1}{4} (2 \ln(1) - 1) = \frac{3e^4 + 1}{4}. \]

In the last step we used the relations: \( \ln(e^2) = 2 \) and \( \ln(1) = 0 \).
Question 2

Let $\mathcal{R}$ be the region in the plane bounded by the curves:

$$y = x^3 \text{ and } y = 9x^2 - 18x.$$ 

Sketch the region $\mathcal{R}$ carefully and find the area of the region $\mathcal{R}$.

The curves meet where both equations hold at once, so where:

$$x^3 = 9x^2 - 18x, \quad x^3 - 9x^2 + 18x = 0,$$

$$0 = x(x^2 - 9x + 18) = x(x - 3)(x - 6),$$

$$x = 0, 3, 6.$$ 

So the meeting points are $(0, 0)$, $(3, 27)$ and $(6, 216)$.

- In the interval $[0, 3]$ the cubic is above the parabola, so the area between the curves in this region is:

$$\int_0^3 (x^3 - (9x^2 - 18x))\,dx = \int_0^3 (x^3 - 9x^2 + 18x)\,dx$$

$$= \left[ \frac{1}{4}x^4 - 3x^3 + 9x^2 \right]_0^3 = \frac{81}{4} - 81 + 81 = \frac{81}{4}. $$

- In the interval $[3, 6]$ the cubic is below the parabola, so the area between the curves in this region is:

$$\int_3^6 (9x^2 - 18x - x^3)\,dx = \left[ 3x^3 - 9x^2 - \frac{1}{4}x^4 \right]_3^6 = 3(216) - 9(36) - \frac{1296}{4} + \frac{81}{4} = \frac{81}{4}. $$

So the required area is $\frac{81}{4} + \frac{81}{4} = \frac{81}{2}$. 


Question 3

Consider the following integral:

\[ \int_0^1 \frac{dt}{(t+1)(t+2)(t+3)}. \]

- Determine the expansion of \( \frac{1}{(t+1)(t+2)(t+3)} \) in partial fractions.

The required decomposition takes the form:

\[ \frac{1}{(t+1)(t+2)(t+3)} = \frac{A}{t+1} + \frac{B}{t+2} + \frac{C}{t+3}. \]

Multiplying both sides by \((t+1)(t+2)(t+3)\) gives the relation:

\[ 1 = A(t+2)(t+3) + B(t+1)(t+3) + C(t+1)(t+2). \]

Put \( t = -1, -2, -3 \), successively:

- When \( t = -1 \), we get: \( 1 = 2A \), so \( A = \frac{1}{2} \).
- When \( t = -2 \), we get: \( 1 = -B \), so \( B = -1 \).
- When \( t = -3 \), we get: \( 1 = 2C \), so \( C = \frac{1}{2} \).

So the required partial fraction decomposition is:

\[ \frac{1}{(t+1)(t+2)(t+3)} = \frac{\frac{1}{2}}{t+1} - \frac{1}{t+2} + \frac{\frac{1}{2}}{t+3}. \]

- Hence evaluate the integral.

The indefinite integral is:

\[ \int \frac{1}{(t+1)(t+2)(t+3)} \, dt = \frac{1}{2} \ln(t+1) - \ln(t+2) + \frac{1}{2} \ln(t+3) + C. \]

\[ = \frac{1}{2} (\ln(t+1) - 2 \ln(t+2) + \ln(t+3)) + C. \]

\[ = \frac{1}{2} \ln \left( \frac{(t+1)(t+3)}{(t+2)^2} \right) + C. \]

So the required definite integral is:

\[ \int_0^1 \frac{1}{(t+1)(t+2)(t+3)} \, dt = \frac{1}{2} \left[ \ln \left( \frac{(t+1)(t+3)}{(t+2)^2} \right) \right]_0^1 = \frac{1}{2} \left( \ln \left( \frac{8}{9} \right) - \ln \left( \frac{3}{4} \right) \right) = \frac{1}{2} \ln \left( \frac{32}{27} \right). \]
Question 4

For each of the following integrals, give an appropriate substitution that will enable the integral to be done. Also give the integral that results from the substitution. You do not need to evaluate the subsequent integral.

- \[ \int \frac{1}{\sqrt{x} + \sqrt[4]{x}} \, dx. \]
  We substitute:
  
  - \( u = \sqrt{x} \), so \( x^4 = u \) and \( \sqrt{x} = u^2 \).
  - Then we have \( du = \frac{4u}{x} \, dx = 4x^3 \, dx \).
  - Then the differential becomes: \[ \frac{1}{\sqrt{x} + \sqrt[4]{x}} \, dx = \frac{1}{u^2 + u} \, du = \frac{4u^2 \, du}{u + 1}. \]

So the integral becomes:

\[
4 \int \frac{u^2}{u + 1} \, du = 4 \int \frac{u^2 - 1 + 1}{u + 1} \, du
= 4 \int (u - 1 + \frac{1}{u + 1}) \, du = 4\left(\frac{u^2}{2} - u + \ln(u + 1)\right) + C
= 2\sqrt{x} - 4\sqrt[4]{x} + 4\ln(\sqrt[4]{x} + 1) + C.
\]

- \[ \int \frac{\cos^3(t)}{\sin^4(t)} \, dt. \]
  We substitute:
  
  - \( u = \sin(t) \), \( du = \cos(t) \, dt \), so \( 1 - u^2 = \cos^2(t) \).
  - \( du = \cos(t) \, dt \).
  - Then the differential becomes:
  
  \[
  \frac{\cos^3(t)}{\sin^4(t)} \, dt = \frac{\cos^2(t)}{\sin^4(t)} \cos(t) \, dt = \frac{1 - u^2}{u^4} \, du = \int \frac{1}{u^4 - u^2} \, du = \int (u^{-4} - u^{-2}) \, du
  \]

Then the integral is:

\[
\int (u^{-4} - u^{-2}) \, du = -\frac{1}{3}u^{-3} + u^{-1} + C = -\frac{1}{3} \csc^3(x) + \csc(x) + C.
\]
\[ \int x^3 e^{-3x^4} \, dx. \]

We substitute:
- \( u = -3x^4 \).
- Then \( du = \frac{du}{dx} \, dx = -12x^3 \, dx \).
- The differential becomes:
  \[ x^3 e^{-3x^4} \, dx = -\frac{1}{12} e^u \, du. \]

So the integral is:
\[
\int -\frac{1}{12} e^u \, du = -\frac{1}{12} e^u + C = -\frac{1}{12} e^{-3x^4} + C.
\]

\[ \int \frac{x^3}{(1 - x^2)^\frac{3}{2}} \, dx. \]

We substitute:
- \( u = 1 - x^2 \), so \( x^2 = 1 - u \).
- \( du = \frac{du}{dx} \, dx = -x \, dx \).
- The differential becomes:
  \[
  \frac{x^3}{(1 - x^2)^\frac{3}{2}} \, dx = \frac{x^2}{(1 - x^2)^\frac{3}{2}} \, (x \, dx) = -\frac{1}{2} \left( \frac{1}{u^2} - \frac{u}{u^2} \right) \, du = -\frac{1}{2} \left( \frac{1}{u^\frac{3}{2}} - \frac{u}{u^\frac{5}{2}} \right) \, du
  \]
  \[ = \frac{1}{2} (u^{-\frac{3}{2}} - u^{-\frac{5}{2}}) \, du \]

So the integral becomes:
\[
\frac{1}{2} \int (u^{-\frac{3}{2}} - u^{-\frac{5}{2}}) \, du = -u^{-\frac{1}{2}} + \frac{1}{3} u^{-\frac{3}{2}} + C
\]
\[ = \frac{1}{3} (1 - x^2)^{-\frac{1}{2}} - (1 - x^2)^{-\frac{3}{2}} + C. \]
Question 5

Estimate the following integral:

\[ \int_{0}^{10} e^{-x^2} dx. \]

- Using the trapezoidal rule with ten intervals.

Also discuss, giving your reasons, whether or not you expect this estimate to be an over-estimate or an under-estimate of the true result.

The intervals have width \( \Delta x = \frac{10-0}{10} = 1 \), so we get:

\[
T_{10} = \frac{1}{2}(f(0)+f(10)+2(f(1)+f(2)+f(3)+f(4)+f(5)+f(6)+f(7)+f(8)+f(9)))
\]

\[
= \frac{1}{2}(1+e^{-100})+e^{-1}+e^{-4}+e^{-9}+e^{-16}+e^{-25}+e^{-36}+e^{-49}+e^{-64}+e^{-81}
\]

\[
= 0.886318602413326.
\]

Plotting the graph of \( f \), we see that the bulk of the integral occurs in the region where the graph is concave down, so that the trapezoids lie below the curve, so we expect that this estimate is an under-estimate.

- Using the midpoint rule with ten intervals.

Also discuss, giving your reasons, whether or not you expect this estimate to be an over-estimate or an under-estimate of the true result.

For ten intervals, we have \( \Delta x = \frac{10-0}{10} = 1 \).

Then we have:

\[
M_{10} = f(\frac{1}{2})+f(\frac{3}{2})+f(\frac{5}{2})+f(\frac{7}{2})+f(\frac{9}{2})+f(\frac{11}{2})+f(\frac{13}{2})+f(\frac{15}{2})+f(\frac{17}{2})+f(\frac{19}{2})
\]

\[
= e^{-\frac{1}{4}}+e^{-\frac{9}{4}}+e^{-\frac{25}{4}}+e^{-\frac{49}{4}}+e^{-\frac{81}{4}}+e^{-\frac{121}{4}}+e^{-\frac{169}{4}}+e^{-\frac{225}{4}}+e^{-\frac{289}{4}}+e^{-\frac{361}{4}}
\]

\[
= 0.886135248492190.
\]

Plotting the graph of \( f \), we see that the bulk of the integral occurs in the region where the graph is concave down, so that the tangent trapezoids lie above the curve, so we expect that this estimate is an over-estimate.
• Using Simpson's rule with twenty intervals.

We can use our results for $T_{10}$ and $M_{10}$:

$$S_{20} = \frac{1}{3}(T_{10} + 2M_{10}) = 0.886196366465902.$$ 

For each estimate, estimate the maximum error:

• $n$ is the number of intervals used in the approximation.

• The integration interval is $[a, b]$, with $b \geq a$.

• The integrand is $f(x)$.

• $K_2$ as the maximum of $|f''(x)|$ on the interval $[a, b]$.

• $K_4$ as the maximum of $|f'''(x)|$ on the interval $[a, b]$.

If $f = e^{-x^2}$, we have:

$$f' = e^{-x^2}(-2x) = -2xf,$$

$$f'' = -2f - 2xf' = -2f - 2x(-2x)f = (4x^2 - 2)f,$$

$$f''' = 8xf + (4x^2 - 2)(-2xf) = (12x - 8x^3)f = -4x(2x^2 - 3)f$$

$$f'''' = (12 - 24x^2)f + (12x - 8x^3)(-2xf) = (12 - 48x^2 + 16x^4)f$$

$$f''' = (-96x + 64x^3)f + (12 - 48x^2 + 16x^4)(-2x)f = -8x(15 - 20x^2 + 4x^4)f.$$ 

For $K_2$, we need to determine $f''(x)$ at its critical points in $(0, 10)$ and at the endpoints $x = 0$ and $x = 10$:

• $f''(0) = -2f(0) = -2$,

• $f''(10) = 398f(10) = 0.1480590238(10^{-40})$,

• The critical point in $(0, 10)$ is where $f'''$ vanishes in $(0, 10)$, so where $2x^2 = 3$, so $x = \sqrt{\frac{3}{2}}$ and then we have:

$$f'' = (4x^2 - 2)e^{-x^2} = (4\left(\frac{3}{2}\right) - 2)e^{-\frac{3}{2}} = 4e^{-\frac{3}{2}} = 0.8925206404.$$ 

So $K_2 = 2$. 

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For $\mathcal{K}_4$, we need to determine $f'''(x)$ at its critical points in $(0, 10)$ and at the endpoints $x = 0$ and $x = 10$: 

- $f'''(0) = 12f(0) = 12$,

- $f''(10) = (12 - 4800 + 160000)f(10) = 0.5774004324(10^{-38})$,

- The critical points in $(0, 10)$ are where $f'''$ vanishes in $(0, 10)$, so where $4x^4 - 20x^2 + 15 = 0$, so where $x^2 = \frac{1}{8}(20 \pm \sqrt{20^2 - 4(4(15))}) = \frac{1}{2}(5 \pm \sqrt{10})$

and then we have:

$$f''' = (12 - 48x^2 + 16x^4)f = (12 - 48x^2 + 4(20x^2 - 15))f$$

$$= (32x^2 - 48) = 16(2 \pm \sqrt{10})e^{-\frac{1}{2}(5 \pm \sqrt{10})}$$

Taking the plus roots, we get:

$$f''' = 16(2 + \sqrt{10})e^{-\frac{1}{2}(5 + \sqrt{10})} = 1.394907048.$$  

Taking the minus roots, we get:

$$f''' = 16(2 - \sqrt{10})e^{-\frac{1}{2}(5 - \sqrt{10})} = -7.419481174.$$  

So $\mathcal{K}_4 = 12$.

Then the required error formulas are:

- For the trapezoidal rule the error estimate for $\mathcal{T}_{10}$ is:

$$\mathcal{E}_T = \frac{\mathcal{K}_2(b - a)^3}{12n^2} = \frac{2(10 - 0)^3}{12(10^2)} = \frac{5}{3}.$$  

- For the midpoint rule the error estimate for $\mathcal{M}_{10}$ is:

$$\mathcal{E}_M = \frac{\mathcal{K}_2(b - a)^3}{24n^2} = \frac{2(10 - 0)^3}{24(10^2)} = \frac{5}{6}.$$  

- For Simpson’s rule the error estimate is:

$$\mathcal{E}_S = \frac{\mathcal{K}_4(b - a)^5}{180n^4} = \frac{12(10 - 0)^5}{180(20^4)} = \frac{1}{24}.$$  

Maple gives the integral numerically as: 0.8862269, so it is easily seen that all our estimates lie well within the allowed errors.
Question 6

A body of mass $m = 10$ kilograms is initially (at time zero) at rest at the origin. The body is acted upon by a force vector $\mathbf{F}$ given as follows, at time $t$ seconds (units are metric), with $0 \leq t < 5$:

$$\mathbf{F} = \left[ \frac{120t}{(\sqrt{25} - t^2)^3}, \ 80 \sin(2t) \right].$$

- Find its acceleration, velocity and position vectors at time $t$.
  
  - The acceleration $\mathbf{a}$ is given by Newton’s Second Law as:
    $$\mathbf{a} = \frac{\mathbf{F}}{m} = \left[ \frac{12t}{(\sqrt{25} - t^2)^3}, \ 8 \sin(2t) \right].$$
  
  - The velocity $\mathbf{v}$ obeys $\frac{d}{dt} \mathbf{v} = \mathbf{a}$, so by FTC, we get:
    $$\mathbf{v} = \int \mathbf{a} \, dt = \int \left[ \frac{12t}{(\sqrt{25} - t^2)^3}, \ 8 \sin(2t) \right] \, dt = [12(25-t^2)^{-\frac{1}{2}}, \ -4 \cos(2t)] + C.$$
    
    Here we used the substitution $u = 25 - t^2$, $du = -2tdt$:
    $$\int \frac{12t}{(\sqrt{25} - t^2)^3} \, dt = \int 12t(25 - t^2)^{-\frac{3}{2}} \, dt = \int -6u^{-\frac{3}{2}} \, du = 12u^{-\frac{1}{2}} + C = 12(25 - t^2)^{-\frac{1}{2}} + C.$$
  
  - Putting $t = 0$ in the velocity formula gives the relation:
    $$0 = [12(25)^{-\frac{1}{2}}, \ -4] + C = \left[ \frac{12}{5}, \ -4 \right] + C,$$
    
    $$C = \left[ -\frac{12}{5}, \ 4 \right].$$
  
    So finally the velocity is given by:
    $$\mathbf{v} = [12(25 - t^2)^{-\frac{1}{2}} - \frac{12}{5}, \ 4 - 4 \cos(2t)]$$
- The position $\mathbf{X}$ obeys $\frac{d}{dt} \mathbf{X} = \mathbf{V}$, so by FTC, we get:

$$\mathbf{X} = \int \mathbf{V} dt = \int \left[ \frac{12}{\sqrt{25 - t^2}} - \frac{12}{5}, \ 4 - 4 \cos(2t) \right] dt$$

$$= [12 \arcsin(t/5) - \frac{12}{5} t, \ 4t - 2 \sin(2t)] + D.$$  

Here we used the substitution $t = 5 \sin(u)$, $dt = 5 \cos(u) du$, $\sqrt{25 - t^2} = \sqrt{25 - 25 \sin^2 u} = 5 \cos(u)$, $u = \arcsin(t/5)$:

$$\int \frac{12}{\sqrt{25 - t^2}} dt = 12 \int \frac{5 \cos(u) du}{5 \cos(u)} = 12 \int du = 12u + C = 12 \arcsin\left(\frac{t}{5}\right) + C.$$  

- Putting $t = 0$ in the position formula gives the relation:

$$0 = [12 \arcsin(0) - \frac{12}{5} (0), \ 4(0) - 2 \sin(2(0))] + D = [0, 0] + D.$$  

So $D = 0$ and the position is given by:

$$\mathbf{X} = [12 \arcsin(t/5) - \frac{12}{5} t, \ 4t - 2 \sin(2t)].$$

- Describe the motion, giving a suitable plot.
  The Maple plot shows that the particle first shoots out in the first quadrant, staying relatively close to the $y$-axis, curving around slightly to the right and reaching the point $[0.522013306, 12.55883100]$ by $t = 3$.
  Then it bends to the left for a short time, before bending around to the right again and flattening out, ending up at the point $[6.84955592, 21.08804222]$ at $t = 5$.

- Find its speed after 4 seconds.
  Putting $t = 4$ in the velocity formula gives:

$$\mathbf{V} = [12(25 - 16)^{-\frac{1}{2}} - \frac{12}{5}, \ 4 - 4 \cos(8)]$$

$$= \left[\frac{8}{5}, \ 4 - 4 \cos(8)\right].$$

The speed in meters per second is then:

$$\sqrt{\left(\frac{8}{5}\right)^2 + (4 - 4 \cos(8))^2} = 4.853321.$$
Question 7

A particle moves in the plane from \( A = [2, -1] \) to \( B = [4, 3] \) along the straight line segment \( AB \).

The particle is acted on by two forces, \( F \) and \( G \) at any point \([x, y]\) of the segment:

\[
F = [4, -2],
\]

\[
G = [2xy^3, 3y^2x^2].
\]

Calculate the work done by each of the forces as the particle moves from \( A \) to \( B \).

- The displacement vector from \( A \) to \( B \) is:

\[
\mathbf{V} = \mathbf{B} - \mathbf{A} = [4, 3] - [2, -1] = [4 - 2, 3 - (-1)] = [2, 4].
\]

So the work \( W_F \) done by the constant force \( F \) is:

\[
W_F = F \cdot V = [4, -2] \cdot [2, 4] = 4(2) + (-2)4 = 8 - 8 = 0.
\]

- Under an infinitesimal displacement \( d\mathbf{X} = [dx, dy] \), the work \( dW \) done by \( G \) is:

\[
dW = G \cdot d\mathbf{X} = [2xy^3, 3y^2x^2] \cdot [dx, dy] = 2xy^3dx + 3y^2x^2dy = xy^2(2ydx + 3xdy).
\]

- We may notice the relation (using the product and power rules):

\[
d(x^2y^3) = y^3d(x^2) + x^2d(y^3) = y^3(2xdx) + x^2(3y^2dy) = dW.
\]

Then the total work done in Joules in going from \( A \) to \( B \) by \( F \) is:

\[
\int_A^B dW = \int_A^B d(x^2y^3) = [x^2y^3]_{[2, -1]}^{[4, 3]} = 4^23^3 - 2^2(-1)^3 = 16(27) + 4 = 436.
\]
Alternatively we first parametrize the line $AB$.
The direction vector is $\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = [4, 3] - [2, -1] = [2, 4]$.
Then the equation of the line is:

$$X = [x, y] = \overrightarrow{A} + t \overrightarrow{AB} = [2, -1] + t[2, 4] = [2 + 2t, -1 + 4t].$$

So we have $x = 2 + 2t$, $y = -1 + 4t$, $dx = \frac{dx}{dt} dt = 2dt$, $dy = \frac{dy}{dt} dt = 4dt$, so $dW$ becomes:

$$dW = xy^2(2ydx + 3xdy) = (2+2t)(-1+4t)^2(2(-1+4t)2dt + 3(2+2t)4dt)$$
$$= 8dt(1+t)(-1+4t)^2(-1+4t+6(1+t)) = 8dt(1+t)(-1+4t)^2(5+10t)$$
$$= 40dt(1 + t)(1 + 2t)(4t - 1)^2 = 40dt(2t^2 + 3t + 1)(16t^2 - 8t + 1)$$
$$= 40dt(32t^4 + 32t^3 - 6t^2 - 5t + 1).$$

The parameter value $t = 0$ gives $A$; the parameter value $t = 1$ gives $B$.
So the total work done by the force $\mathbf{F}$ in Joules is:

$$\int_A^B dW = \int_0^1 40(32t^4 + 32t^3 - 6t^2 - 5t + 1)dt$$
$$= 40\left[\frac{32t^5}{5} + 8t^4 - 2t^3 - \frac{5t^2}{2} + t\right]_0^1 = 40\left(\frac{32}{5} + 8 - 2 - \frac{5}{2} + 1\right)$$
$$= 256 + 320 - 80 - 100 + 40 = 436.$$

Alternatively, we see that the line $AB$ has slope $\frac{3-(-1)}{4-2} = \frac{4}{2} = 2$.
So the line $AB$ has equation $y - 3 = 2(x - 4)$, or $y = 2x - 5$.
The variable $x$ ranges from $x = 2$ to $x = 4$; also $dy = \frac{dy}{dx}dx = 2dx$, so:

$$dW = xy^2(2ydx + 3xdy) = x(2x - 5)^2(2(2x - 5)dx + 3x(2dx))$$
$$= x(2x - 5)^2dx(10x - 10)$$
$$= 10dx(x - 1)(4x^2 - 20x + 25) = 10dx(4x^3 - 24x^2 + 45x - 25)$$
$$= 10dx(4x^4 - 24x^3 + 45x^2 - 25x).$$

Then the total work done by the force $\mathbf{F}$ in Joules is:

$$\int_A^B dW = \int_2^4 (40x^4 - 240x^3 + 450x^2 - 250x)dx$$
$$= \left[8x^5 - 60x^4 + 150x^3 - 125x^2\right]_2^4 = \left[x^2(8x^3 - 60x^2 + 150x - 125)\right]_2^4$$
$$= 4^2(8(64) - 60(16) + 150(4) - 125) - 2^2(8(8) - 60(4) + 150(2) - 125)$$
$$= 16(512 - 960 + 600 - 125) - 4(64 - 240 + 300 - 125)$$
$$= 16(27) - 4(-1) = 432 + 4 = 436.$