Differential Geometry I, Fall 2004
Homework Assignments
Homework 1, due Thursday September 16th

Do the following problems:

- Consider the $n$-sphere, $S^n$, with defining equation in $\mathbb{R}^{n+1}$, $\bar{x} \cdot x + z^2 = 1$, where $z$ is real and $x \in \mathbb{R}^n$.
  - Using stereographic projection to the equatorial plane, cover the sphere in two open sets $U_1$ and $U_2$, which are the complements of the North pole ($z = 1$) and South pole ($z = -1$), respectively, equipped with homeomorphisms $\phi_1 : U_1 \to \mathbb{R}^n$ and $\phi_2 : U_2 \to \mathbb{R}^n$ and obtain explicit formulas for the inverses of each of these maps. Include a proof that these maps are homeomorphisms.
  - Give appropriate domains for the compositions $\phi_{12} = \phi_2 \circ \phi_1^{-1}$ and $\phi_{21} = \phi_1 \circ \phi_2^{-1}$ and compute these compositions. Show that these compositions are real analytic, so give the sphere a real analytic differential structure.
  - Give the definition of real analytic functions on the sphere relative to this differential structure and show that the restriction to the sphere of each of the Euclidean co-ordinate functions gives a real analytic function.

- For the case of the two-sphere, show that the two real co-ordinates for each patch may be combined appropriately into a complex number, such that the patching relation is complex analytic.

- For the case of the four-sphere, show that the four real co-ordinates for each patch may be combined appropriately into a quaternion, such that the patching relation expressed in the language of quaternions is as simple as in the complex case.

- Is there an analogous description of the eight-sphere, using octonions?

Homework 2, due Wednesday September 29th

Do the following problems:

- Obtain a description of the tangent bundle of the $n$-sphere in terms of co-ordinates.
  You may regard the tangent bundle as being given as the set of all pairs $(\bar{x}, \bar{p})$ in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, with $\bar{x} \cdot x = 1$ and $\bar{x} \cdot \bar{p} = 0$. 
• Prove that every sheaf over a manifold is itself a manifold in a natural way (though not necessarily Hausdorff).

• Let $\mathbb{R}^+$ denote the real line with double origin, so $\mathbb{R}^+$ is the quotient space of the disjoint union of two copies of the reals $\mathbb{R}^+ = \mathbb{R} \times \{1\}$ and $\mathbb{R}^- = \mathbb{R} \times \{-1\}$ in $\mathbb{R}^2$ by the identification $(x, 1) \in \mathbb{R}^+$ is identified with $(y, -1) \in \mathbb{R}^-$, whenever $x = y \neq 0$, with the quotient topology. Prove that $\mathbb{R}^+$ is a (non-Hausdorff) manifold and is naturally a sheaf over the reals.

• Give, with proof, a patching for the space of all (affine) straight lines in $\mathbb{R}^2$.

• Give, with proof, a patching for the space of all (affine) straight lines in $\mathbb{R}^3$.

**Homework 3, due Wednesday October 13th**

Do the following problems:

• For the vector field $x^2 \frac{d}{dx}$ in $\mathbb{R}^1$, verify directly that the resulting motion has the partial group property.

• Consider the vector field $(a \times \vec{x}).\partial_{\vec{x}}$ in Euclidean three-space. Here $\vec{x}$ is the position vector of a point and $a$ is a fixed non-zero vector. Integrate the vector field explicitly and show that the resulting motion gives a one-parameter group of Euclidean rotations in space, with axis represented by the direction of the vector $\vec{a}$. Identify the rotational period. Also identify the invariant submanifolds of the flow.

• Construct a vector field on the two-dimensional sphere which vanishes at only one point and obtain its flow.

• Construct an everywhere non-zero vector field on the three-dimensional sphere and identify its flow.

**Homework 4, due Wednesday October 27th**

Do the following problems:
• For the vector fields $L_1 = x\partial_x$, $L_2 = y\partial_x$, $L_3 = x\partial_y$ and $L_4 = y\partial_y$ in $\mathbb{R}^2$:

  – Find the Lie brackets $[L_i, L_j]$ and show that the $L_i$ span a four-dimensional Lie algebra $L$, say; also find the structure constants of the Lie algebra $L$.

  – Find with proof a three-dimensional subalgebra $M$ of $L$ with the property that $[M, M] = M$.

    Discuss whether or not $M$ is unique.

• Consider the span $J$ of the collection of vector fields of the form $L_\mathbf{a} = (\mathbf{a} \times \mathbf{z}).\partial_\mathbf{z}$ in Euclidean three-space.

  Here $\mathbf{z}$ is the position vector of a point and $\mathbf{a}$ is a constant vector.

  – Compute the Lie bracket $[L_\mathbf{a}, L_\mathbf{b}]$ and hence show that $J$ is a Lie algebra.


  – Show that every element of $J$ is tangent to each sphere, centered at the origin.

• Show that the collection $K$ of all vector fields on $\mathbb{R}^3$ that annihilate a given smooth function $f(x, y, z)$ on $\mathbb{R}^3$ form a Lie algebra over the reals.

  What is the geometrical interpretation of this Lie algebra?

  Describe this collection $K$ for the case of the function $x^2 + y^2 - z^2$. 

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