Integrated Calculus I Practice for the Final: A Baker’s Dozen of Solutions 12/7/4

Question 1

Determine the derivative of each of the following functions:

- \( a(x) = (x^3 + 1)^4 \sin^2(2x + 1) \).
  \[ a'(x) = 4(x^3+1)^3(3x^2) \sin^2(2x+1)+(x^3+1)^4(2 \sin(2x+1) \cos(2x+1))(2). \]

- \( b(x) = \frac{\sqrt{x} - 1}{\sqrt{x} - 1} \).
  \[ b(x) = \frac{x^{\frac{1}{2}} - 1}{x^{\frac{3}{2}} - 1}, \quad b'(x) = \frac{\frac{1}{2}x^{-\frac{1}{2}}(x^{\frac{1}{2}} - 1) - \frac{1}{3}x^{-\frac{3}{2}}(x^{\frac{3}{2}} - 1)}{(x^{\frac{3}{2}} - 1)^2}. \]

- \( c(x) = \tan^2(2 \ln(x)) + e^{4 \cos(x)} \).
  \[ c'(x) = 2 \tan(2 \ln(x)) \sec^2(2 \ln(x)) \left( \frac{2}{x} \right) + e^{4 \cos(x)}(-4 \sin(x)). \]

- \( d(x) = x^{(x^2)} \).
  \[ \ln(d) = x^2 \ln(x), \quad \frac{d'}{d} = 2x \ln(x) + x^2 \left( \frac{1}{x} \right) = x(2 \ln(x) + 1), \]
  \[ d' = xd(2 \ln(x) + 1) = x(x^{(x^2)})(2 \ln(x) + 1) = x^{(1+x^2)}(2 \ln(x) + 1). \]
  Alternatively we may write: \( d = x^{(x^2)} = (e^{\ln(x)})(x^2) = e^{x^2 \ln(x)} \), which gives \( d' = e^{x^2 \ln(x)} \left( 2x \ln(x) + x^2 \left( \frac{1}{x} \right) \right) \).

- \( e(x) = p(q(x)) \), where \( p'(t) = t^3 - t \) and \( q'(x) = \sin(x) \).
  By the chain rule, we have \( e'(x) = p'(q(x))q'(x) = ((q(x))^3 - q(x)) \sin(x) \).
  Note that since \( q'(x) = \sin(x) \), we know that \( q(x) = -\cos(x) + C \), where \( C \) is constant (but not specified in the problem).
  So we may also write: \( e'(x) = ((-\cos(x) + C)^3 - (-\cos(x) + C)) \sin(x) \).

- \( f(x) = \int_{\sin(x)}^{\cos(x)} \frac{1}{t^2 + 1} dt \).
  \[ f'(x) = \frac{1}{\cos^2(x) + 1}(- \sin(x)) - \frac{1}{\sin^2(x) + 1}(\cos(x)). \]
Question 2

Determine each of the following indefinite integrals:

• \( A = \int \left( x^3 - x^{\frac{3}{2}} - \frac{1}{x} + \frac{1}{\sqrt{x}} + \frac{1}{x^2} \right) \, dx \)
  
  \[ A = \frac{x^4}{4} - \frac{2x^{\frac{7}{2}}}{7} - \ln(Gx) + 2x^{\frac{1}{2}} - x^{-1}. \]

• \( B = \int (\sin(2x) - \cos(3x) + \sec^2(4x) - e^{5x}) \, dx. \)
  
  \[ B = -\frac{1}{2} \cos(2x) - \frac{1}{3} \sin(3x) + \frac{1}{4} \tan(4x) - \frac{1}{5} e^{5x} + H. \]

• \( C = \int \left( xe^{x^2+1} - \frac{x}{x^2+1} + \frac{2x}{(x^2+1)^3} \right) \, dx. \)
  
  Put \( u = x^2 + 1, \) so \( du = \frac{du}{dx} \, dx = 2 \, dx. \) Then we have:
  
  \[ C = \frac{1}{2} \int \left( e^u - \frac{1}{u} + \frac{2}{u^3} \right) \, du = \frac{1}{2} (e^{x^2+1} - \ln(J(x^2 + 1)) - (x^2 + 1)^{-2}). \]

• \( D = \int \left( \frac{(\ln(x))^{999}}{x} + \frac{1}{x(\ln(x))^{1001}} \right) \, dx. \)
  
  Put \( u = \ln(x), \) so \( du = \frac{du}{dx} \, dx = \frac{1}{x} \, dx. \) Then we have:
  
  \[ D = \int (u^{999} + u^{-1001}) \, du = \frac{u^{1000} - u^{-1000}}{1000} + K = \frac{(\ln(x))^{1000} - (\ln(x))^{-1000}}{1000} + K. \]
\[ E = \int \left( \frac{x - 1}{\sqrt{x + 1}} + \frac{x^2 + 1}{x + 1} \right) \, dx. \]

Put \( u = x + 1 \), so \( du = \frac{dx}{x + 1} \) and \( x = u - 1 \).

Then we have:

\[ E = \int \left( \frac{u - 1 - 1}{\sqrt{u}} + \frac{(u - 1)^2 + 1}{u} \right) \, du \]

\[ = \int \left( \frac{u - 2}{\sqrt{u}} + \frac{u^2 - 2u + 1 + 1}{u} \right) \, du \]

\[ = \int \left( \frac{u}{u^{\frac{1}{2}}} - \frac{2}{u^{\frac{1}{2}}} + \frac{u^2 - 2u + 2}{u} \right) \, du \]

\[ = \int \left( \frac{u^{\frac{1}{2}}}{u^{\frac{1}{2}}} - 2u^{-\frac{1}{2}} + u - 2 + \frac{2}{u} \right) \, du \]

\[ = \frac{3}{5} u^{\frac{3}{2}} - 3u^{\frac{3}{2}} + \frac{u^2}{2} - 2u + 2 \ln(Lu) \]

\[ = \frac{3}{5}(x + 1)^{\frac{3}{2}} - 3(x + 1)^{\frac{3}{2}} + \frac{(x + 1)^2}{2} - 2(x + 1) + 2 \ln(L(x + 1)). \]

\[ F = \int \left( \frac{3}{1 + t^2} - \frac{4t}{t^2 + 1} + \frac{5}{\sqrt{1 - t^2}} - \frac{6t}{\sqrt{1 - t^2}} \right) \, dt \]

\[ F = F_1 + F_2 + F_3 + F_4. \]

\[ F_1 = \int \frac{3}{1 + t^2} \, dt = 3 \arctan(t) + M_1. \]

\[ F_2 = \int -\frac{4t}{t^2 + 1} \, dt = -2 \int \frac{du}{u} = -2 \ln(M_2u) = -2 \ln(M_2(t^2 + 1)). \]

Here we made the substitution \( u = t^2 + 1 \), \( du = \frac{dt}{2} \).

\[ F_3 = \int \frac{5}{\sqrt{1 - t^2}} \, dt = 5 \arcsin(t) + M_3. \]

\[ F_4 = \int -\frac{6t}{\sqrt{1 - t^2}} \, dt = \int \frac{3du}{u^{\frac{3}{2}}} = 6u^{\frac{1}{2}} + M_4 = 6\sqrt{1 - t^2} + M_4. \]

Here we made the substitution \( u = 1 - t^2 \), \( du = \frac{dt}{2} \).

So we get:

\[ F = 3 \arctan(t) - 2 \ln(M(t^2 + 1)) + 5 \arcsin(t) + 6\sqrt{1 - t^2}. \]

Here \( G, J, H, K, L, M, M_1, M_2, M_3 \) and \( M_4 \) are all constant.
Question 3

Find the area between the curves $y = x^2 - 3x$ and $y = 3 + 2x - x^2$ (begin by sketching these curves).

Each curve is a parabola, one downward pointing and the other upward pointing. They meet where both equations hold at once, so where:

\[ x^2 - 3x = 3 + 2x - x^2, \]
\[ 2x^2 - 5x - 3 = 0, \]
\[ (2x + 1)(x - 3) = 0, \]
\[ x = -\frac{1}{2}, \quad 3. \]

When $x = -\frac{1}{2}$, $y = (-\frac{1}{2})^2 - 3(-\frac{1}{2}) = \frac{1}{4} + \frac{3}{2} = \frac{7}{4}$, for both curves.

When $x = 3$, $y = 3^2 - 3(3) = 0$, for both curves.

So the curves meet at $B = (-\frac{1}{2}, \frac{7}{4})$ and at $C = (3, 0)$.

Then the area in question is the region enclosed by the curves between $B$ and $C$.

In the region between $B$ and $C$, the second parabola is the higher, so, using vertical strips, we have the area element $dA = (y_+ - y_-)dx$, where $y_+ = 3 + 2x - x^2$ and $y_- = x^2 - 3x$, which gives:

\[ dA = (y_+ - y_-)dx = (3 + 2x - x^2 - (x^2 - 3x))dx = (3 + 5x - 2x^2)dx. \]

Then the area $A$ enclosed by the curves is:

\[
A = \int_{-\frac{1}{2}}^{3} (3 + 5x - 2x^2)dx
\]

\[ = \left[3x + \frac{5}{2}x^2 - \frac{2}{3}x^3\right]_{-\frac{1}{2}}^{3}
\]

\[ = 3(3) + \frac{5}{2}(3)^2 - \frac{2}{3}(3)^3 - \left(3\left(-\frac{1}{2}\right) + \frac{5}{2}\left(-\frac{1}{2}\right)^2 - \frac{2}{3}\left(-\frac{1}{2}\right)^3\right)
\]

\[ = 9 + \frac{45}{2} - 18 - \left(-\frac{3}{2} + \frac{5}{8} + \frac{1}{12}\right)
\]

\[ = \frac{27}{2} - \left(\frac{1}{24}\right)(-36 + 15 + 2) = \frac{27}{2} + \frac{19}{24}
\]

\[ = \frac{1}{24}(324 + 19) = \frac{343}{24}. \]
Question 4

Find the area between the curves \( x = 4y - 3 \) and \( x = y^2 \) (begin by sketching these curves).

The parabola \( x = y^2 \) meets the straight line \( x = 4y - 3 \) where both equations hold at once, so where:

\[
4y - 3 = y^2, \\
y^2 - 4y + 3 = 0, \\
(y - 1)(y - 3) = 0, \\
y = 1, 3, \\
(x, y) = (1, 1), \quad (x, y) = (9, 3).
\]

So the curves meet at \( B = (1, 1) \) and at \( C = (9, 3) \).

Then the area in question is the region enclosed by the curves between \( B \) and \( C \).

In the region between \( B \) and \( C \), the straight line is to the right, so, using horizontal strips, we have the area element \( dA = (x_+ - x_-)dy \), where \( x_+ = 4y - 3 \) and \( x_- = y^2 \), which gives:

\[
dA = (x_+ - x_-)dy = (4y - 3 - y^2)dy.
\]

Then the area \( A \) enclosed by the curves is:

\[
A = \int_B^C dA = \int_1^3 (4y - 3 - y^2)dy = \left[ 2y^2 - 3y - \frac{1}{3}y^3 \right]_1^3
\]

\[
= (2(3)^2 - 3(3) - \frac{1}{3}(3)^3) - (2(1)^2 - 3(1) - \frac{1}{3}(1)^3) = 18 - 9 - 9 - (2 - 3 - \frac{1}{3}) = \frac{4}{3}.
\]

Note that alternatively, we can use vertical strips here, with the area element:

\[
dA = (y_+ - y_-)dx = (\sqrt{x} - \frac{x + 3}{4})dx.
\]

Here \( x = y_+^2 \), with \( y_+ \geq 0 \), so \( y_+ = \sqrt{x} \) and \( x = 4y_+ - 3 \), so \( y_- = \frac{x + 3}{4} \).

Then the area integral is:

\[
A = \int_B^C dA = \int_1^9 (\sqrt{x} - \frac{x + 3}{4})dx = \left[ \frac{2x^{\frac{3}{2}}}{3} - \frac{1}{8}(x + 3)^2 \right]_1^9
\]

\[
= \frac{2(9)^{\frac{3}{2}}}{3} - \frac{1}{8}(9 + 3)^2 - (\frac{2(1)^{\frac{3}{2}}}{3} - \frac{1}{8}(1 + 3)^2) = 18 - 18 - (\frac{2}{3} - 2) = \frac{4}{3}.
\]
Question 5

Use the trapezoidal rule with five intervals to estimate numerically the integral \( J = \int_0^1 \frac{4}{t^2 + 1} \, dt \) and sketch the graph of the function \( \frac{4}{t^2 + 1} \) and the relevant trapezoids.

Also determine \( K_2 \), the maximum of the absolute value of the second derivative of the function \( \frac{4}{t^2 + 1} \) on the interval \([0, 1]\) and hence find the maximum possible error in your estimate of the integral.

Explain why your estimates lead to a numerical estimation of the number \( \pi \) and whether or not you can determine if your estimate is too high or too low.

Put \( f(t) = \frac{4}{t^2 + 1} \).

The trapezoidal rule with \( n = 5 \) for the integral \( J \) has intervals of width \( \frac{1}{5} \).

The left Riemann sum \( L_5 \) is then:

\[
L_5 = \frac{1}{5} \left( f(0) + f\left(\frac{1}{5}\right) + f\left(\frac{2}{5}\right) + f\left(\frac{3}{5}\right) + f\left(\frac{4}{5}\right) \right)
\]

\[
= \frac{1}{5} \left( 4 + \frac{4}{1 + \left(\frac{1}{5}\right)^2} + \frac{4}{1 + \left(\frac{2}{5}\right)^2} + \frac{4}{1 + \left(\frac{3}{5}\right)^2} + \frac{4}{1 + \left(\frac{4}{5}\right)^2} \right)
\]

\[
= \frac{1}{5} \left( 4 + \frac{100}{25 + 1} + \frac{100}{25 + 4} + \frac{100}{25 + 9} + \frac{100}{25 + 16} \right)
\]

\[
= 20 \left( \frac{1}{25} + \frac{1}{26} + \frac{1}{29} + \frac{1}{34} + \frac{1}{41} \right) = \frac{4381576}{1313845} = 3.334926114.
\]

The right Riemann sum \( R_5 \) is then:

\[
R_5 = \frac{1}{5} \left( f\left(\frac{1}{5}\right) + f\left(\frac{2}{5}\right) + f\left(\frac{3}{5}\right) + f\left(\frac{4}{5}\right) + f(1) \right)
\]

\[
= L_5 - \frac{f(0)}{5} + \frac{f(1)}{5} = L_5 - \frac{4}{5} + \frac{2}{5} = L_5 - \frac{2}{5} = \frac{3856038}{1313845} = 2.934926114.
\]

Then the trapezoidal estimate for \( J \) with 5 intervals is:

\[
T_5 = \frac{1}{2} (L_5 + R_5) = \frac{1}{2} \left( \frac{4381576 + 3856038}{1313845} \right) = \frac{4118807}{1313845} = 3.134926114.
\]
We compute derivatives:
\[
f'(t) = -\frac{4(2t)}{(t^2 + 1)^2} = -\frac{8t}{(t^2 + 1)^2}, \quad f''(t) = -\frac{8}{(t^2 + 1)^2} - \frac{8t(-2(2t))}{(t^2 + 1)^3} = -\frac{8(t^2 + 1)}{(t^2 + 1)^3} + \frac{32t^2}{(t^2 + 1)^3} = \frac{32t^2 - 8t^2 - 8}{(t^2 + 1)^3} = \frac{8(3t^2 - 1)}{(t^2 + 1)^3}.
\]
Plotting \(|f''(t)|\) on \([0, 1]\), we see that it attains its maximum of 8 at \(t = 0\). So \(K_2 = 8\). Alternatively, we compute one more derivative:
\[
f'''(t) = \frac{8(6t)}{(t^2 + 1)^3} - \frac{8(3t^2 - 1)(3(2t))}{(t^2 + 1)^4} = \frac{48t}{(t^2 + 1)^4}(t^2 + 1 - (3t^2 - 1)) = \frac{48t}{(t^2 + 1)^4}(2 - 2t^2) = \frac{96t(1-t^2)}{(t^2 + 1)^4}.
\]
On the interval \((0, 1)\), the factors \(t, 1 - t^2, \) and \(t^2 + 1\) are all positive, so \(f'''(t) > 0\) on \((0, 1)\), so \(f''(t)\) is increasing on \([0, 1]\).
So the maximum of \(|f''(t)|\) occurs at either \(t = 0\) or \(t = 1\).
Putting \(t = 0\), we get \(|f''(0)| = 8\) and putting \(t = 1\), we get \(|f''(1)| = 2\).
So \(K_3 = 8\), as before. Then with \(n = 5\), \(b = 1\) and \(a = 0\), the trapezoidal error formula: \(E_T = \frac{K_3(b-a)^3}{12n^2}\) gives the maximal error as:
\[
E_T = \frac{8(1-0)^3}{12(5)^2} = \frac{8}{300} = \frac{2}{75} = 0.2666.
\]
We cannot easily say whether or not the estimate is too high or too low, because the curve \(y = \frac{4}{1+t^2} = f(t)\) has an inflection point at \(t = \frac{1}{\sqrt{3}} = 0.5773502693\) in the interval \([0, 1]\).
To the left of the inflection point, \(f''(t) < 0\), so the graph of \(f(t)\) is concave down and the trapezoids are below the curve, but to the right, they are above, since the curve is concave up.
By looking at the trapezoids, it appears that the ones to the left of the inflection point are more below the curve than the ones to the right are above, so the net estimate is probably below the true value.
Finally by FTC, we have \(J = \int_0^1 \frac{4}{1+t^2}dt = [4\arctan(t)]_0^1 = 4(\arctan(1) - \arctan(0)) = 4(\frac{\pi}{4} - 0) = \pi\).
So the exact value of the integral is \(\pi = 3.141592654\) and \(T_5 = 3.134926114\) gives an approximation to its value, slightly low, as expected.
The actual error is about 0.00666654, well within our estimation of the maximal allowed error.
The percentage error is about 0.21220257 percent.
Question 6

A particle of mass 4 kilos moving in the plane is acted on by a force:
\[ \mathbf{F} = \left[ 48t, \frac{24}{(t+1)^2} \right] \] at time \( t \) seconds.

If the particle is initially at rest at the origin, where is the particle at time 2
seconds and how fast is it going then?

By Newton’s Second Law, the acceleration \( \mathbf{A} \) is given by:
\[ \mathbf{A} = \frac{1}{m} \mathbf{F} = \frac{1}{4} \left[ 48t, \frac{24}{(t+1)^2} \right] = \left[ 12t, 6(t+1)^{-2} \right]. \]

Then for the velocity \( \mathbf{V} \), since \( \mathbf{A} = \frac{d\mathbf{V}}{dt} \), we have:
\[ \mathbf{V} = \int \mathbf{A} dt = \int \left[ 12t, 6(t+1)^{-2} \right] dt = \left[ 6t^2, -6(t+1)^{-1} \right] + \mathbf{C}. \]

Putting \( t = 0 \), we get: \( [0, 0] = [0, -6] + \mathbf{C} \), since initially the particle is at rest.
So \( \mathbf{C} = -[0, -6] = [0, 6] \), giving: \( \mathbf{V} = [6t^2, 6 - 6(t+1)^{-1}] \).

Next for the position \( \mathbf{X} \), since \( \mathbf{V} = \frac{d\mathbf{X}}{dt} \), we have:
\[ \mathbf{X} = \int \mathbf{V} dt = \int \left[ 6t^2, 6 - 6(t+1)^{-1} \right] dt = \left[ 2t^3, 6t - 6 \ln(t+1) \right] + \mathbf{D}. \]

Putting \( t = 0 \), we get: \( [0, 0] = [0, 0] + \mathbf{D} \), since initially the particle is at the origin.
So \( \mathbf{D} = [0, 0] \) and \( \mathbf{X} = [2t^3, 6t - 6 \ln(t+1)] \).

When \( t = 2 \), we find:
\[ \mathbf{X} = [2(2)^3, 6(2) - 6 \ln(2 + 1)] = [16, 12 - 6 \ln(3)] = [16, 5.408326266], \]
\[ \mathbf{V} = [6(2)^2, 6 - 6(2+1)^{-1}] = [24, 6 - \frac{6}{3}] = [24, 4]. \]

The speed of the particle is then \( |\mathbf{V}| = \sqrt{24^2 + 4^2} = 4\sqrt{6^2 + 1} = 4\sqrt{37} = 24.33105012 \).

So at time \( t = 2 \), the particle is at \( [16, 12 - 6 \ln(3)] = [16, 5.408326266] \)
and has speed \( 4\sqrt{37} = 24.33105012 \) meters per second.
Question 7

Find the linear approximation to the function \( f(t) = e^{2t} \), valid near the point with \( t = 2 \).

Sketch the graphs of the function and your linear approximation on the interval \([1, 3]\).

Use your linear approximation to estimate the quantity \( f(2.25) \) and compare your estimate with the true value.

The linear approximation \( f_1(t) \) to \( f(t) \) based at \( t = a \) is:

\[ f_1(t) = f(a) + f'(a)(t - a). \]

Here we have \( a = 2, \, f(t) = e^{2t}, \, f'(t) = 2e^{2t}, \, f(a) = f(2) = e^{2(2)} = e^4 \) and \( f''(a) = f''(2) = 2e^{2(2)} = 2e^4 \), so we get:

\[ f_1(t) = f(2) + f'(2)(t - 2) = e^4 + 2e^4(t - 2) = e^4(1 + 2t - 4) = e^4(2t - 3). \]

The graphs show that the linear approximation is only accurate quite near to \( t = 2 \).

Also the graph of \( f(t) \) is everywhere concave up, since \( f''(t) = 4e^{2t} > 0 \), so the linear approximation is an underestimate.

We have at \( t = 2.25 \),

\[ f_1(2.25) = e^4(2(2.25) - 3) = e^4(1.5) = 81.89722504, \]

\[ f(2.25) = e^{2(2.25)} = e^{4.5} = 90.01713130. \]

The linear estimate is low, as expected and the percentage error is quite large:

\[ \frac{100(e^{4.5} - e^4(1.5))}{e^{4.5}} = 100 - 150e^{-0.5} = 9.02040104. \]
Question 8

Solve the following differential equations and for each determine the value of the solution $y$ at $t = 4$.

- $\frac{dy}{dt} = \frac{4t^3}{3y^2}, \quad y(1) = 4.$
  
  We separate and integrate:
  
  $3y^2 \, dy = 4t^3 \, dt, \quad \int 3y^2 \, dy = \int 4t^3 \, dt, \quad y^3 = t^4 + C.$
  
  Putting $t = 1$ and $y = 4$, we get:
  
  $4^3 = 1 + C, \quad C = 4^3 - 1 = 64 - 1 = 63, \quad y^3 = t^4 + 63,$
  
  $y = \sqrt[3]{t^4 + 63}.$
  
  Putting $t = 4$, we get:
  
  $y = \sqrt[3]{4^4 + 63} = \sqrt[3]{256 + 63} = \sqrt[3]{319} = 6.832771452.$

- $\frac{dy}{dt} = \frac{(y^2 + 1)^2}{8y\sqrt{t+1}}, \quad y(3) = 1.$
  
  We separate and integrate:
  
  $8y \, dy = \frac{dt}{\sqrt{t+1}}, \quad \int \frac{8y \, dy}{(y^2 + 1)^2} = \int \frac{dt}{\sqrt{t+1}}, \quad -4(y^2 + 1)^{-1} = 2\sqrt{t+1} + D.$
  
  Putting $t = 3$ and $y = 1$, we get:
  
  $-4(1^2 + 1)^{-1} = 2\sqrt{3+1} + D, \quad \frac{-4}{2} = 2(2) + D, \quad D = -2 - 4 = -6,$
  
  $-4(y^2 + 1)^{-1} = 2\sqrt{t+1} - 6, \quad (y^2 + 1)^{-1} = \frac{1}{2}(3 - \sqrt{t+1}),$
  
  $y^2 + 1 = \frac{2}{3 - \sqrt{t+1}} = \frac{2(3 + \sqrt{t+1})}{9 - (t+1)} = \frac{6 + 2\sqrt{t+1}}{8 - t},$
  
  $y = \sqrt[3]{\frac{6 + 2\sqrt{t+1}}{8 - t} - 1} = \sqrt[3]{\frac{6 + 2\sqrt{t+1} - 8 + t}{8 - t}} = \sqrt[3]{\frac{2\sqrt{t+1} + t - 2}{8 - t}}.$
  
  Putting $t = 4$, we get: $y = \sqrt[3]{\frac{\sqrt{6} + 5}{2}} = 1.272019650.$
Question 9

A ladder is sliding down a vertical wall.
The ladder is 40 feet long.
The upper end is moving downward along the ground from the wall at 6 feet per second.
How fast is the lower end sliding away from the wall along the (horizontal) ground, when it is 24 feet from the wall?

Let the upper end be at \((0; y)\) and the lower end at \((x; 0)\).
Then we have \(x^2 + y^2 = 40^2 = 1600\), by Pythagoras’ Theorem.
Differentiating this relation with respect to time, we get:

\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0.
\]

At the time in question, we have:
- \(x = 24\),
- \(y = \sqrt{1600 - 24^2} = \sqrt{1600 - 576} = \sqrt{1024} = 32\),
- \(\frac{dy}{dt} = -6\).

Then we have:

\[
2(24) \frac{dx}{dt} + 2(32)(-6) = 0,
\]

\[
48 \frac{dx}{dt} - 384 = 0,
\]

\[
\frac{dx}{dt} = \frac{384}{48} = 8.
\]

So at the time in question, the lower end of the ladder is moving outwards at 8 feet per second.
Question 10

A painting of mass 30 kilograms is suspended five meters below a horizontal ceiling on the wall of an art gallery by two wires. Taking the origin directly above the painting the painting is attached to the wires at the point $A = [0, -5]$. Then the wires are attached to the ceiling at $B = [10, 0]$ and $C = [-5, 0]$. Give a sketch of the system of wires.

Find the length of each wire and the angles of the triangle $ABC$.

Find the forces in the wires.

We have:

- $AB = B - A = [10, 0] - [0, -5] = [10, 5]$.
- $AC = C - A = [-5, 0] - [0, -5] = [-5, 5]$.
- The force $F$ in the wire $AB$ is $F = sAB = s[10, 5] = [10s, 5s]$, for some scalar $s$.
- The force $G$ in the wire $AC$ is $G = tAC = t[-5, 5] = [-5t, 5t]$, for some scalar $t$.
- The gravitational force due to the action of gravity on the painting at the point of attachment is $[0, -30g]$.

The total force at the point of attachment is zero, giving the equation:

$$0 = F + G + [0, -30g],$$

$$[0, 0] = [10s, 5s] + [-5t, 5t] + [0, -30g],$$

$$[0, 0] = [10s - 5t, 5s + 5t - 30g],$$

$$10s - 5t = 0, \quad 5s + 5t - 30g = 0,$$

$$t = 2s, \quad s + t = 6g,$$

$$3s = 6g, \quad s = 2g, \quad t = 4g,$$

$$F = [10s, 5s] = [20, 10]g, \quad |F| = g\sqrt{20^2 + 10^2} = 10g\sqrt{5} = 22.36067977g = 219.1346617,$$

$$G = [-5t, 5t] = [-20, 20]g, \quad |G| = g\sqrt{(-20)^2 + 20^2} = 20g\sqrt{2} = 28.28427124g = 277.1858582.$$

Note that this solution obeys: $F + G = [20, 10]g + [-20, 20]g = [0, 30g]$, as required.

So the force in the wire $AB$ has size 219.1346617 Newtons, whereas the force in the wire $AC$ has size 277.1858582 Newtons,
Question 11

An aluminum paint can is to hold a volume of 120 cubic inches of paint. Find the dimensions of the can that will minimize the amount of aluminum used in its construction.

You may assume that the aluminum is very thin, but that the material for the top and base is twice as thick as that used for the cylindrical sides.

Let the radius of the base and top be \( x \) inches.
Let the height of the cylindrical sides be \( y \) inches.

Then the area of the base and the top is \( 2\pi x^2 \) square inches.
The area of the cylindrical sides is \( 2\pi xy \) square inches.
If the thickness of the sides is \( h \) inches, then that of the base and top is \( 2h \) inches.
So the amount of material used in the base and top is \( 2\pi x^2(2h) = 4\pi x^2h \) cubic inches.
The amount used in the sides is \( 2\pi xyh \) cubic inches.
So the total amount of material used is \( A = (2\pi xy + 4\pi x^2)h = 2h(\pi xy + 2\pi x^2) \) cubic inches.
The volume of the can is \( \pi x^2y = 120 \), so \( y = \frac{120}{\pi x^2} \).
Substituting into the formula for \( A \), we get:

\[
A = 2h(\pi \frac{120}{x^2} + 2\pi x^2) = 2h(\frac{120}{x} + 2\pi x^2).
\]

As \( x \to 0^+ \) and as \( x \to \infty \), we have \( A \to \infty \).
So \( A \) must have an absolute minimum at a critical point on the interval \((0, \infty)\).
Differentiating, we get:

\[
A' = 2h(-\frac{120}{x^2} + 4\pi x) = \frac{8h}{x^2}(\pi x^3 - 30).
\]
This vanishes only where \( \pi x^3 = 30 \), or \( x^3 = \frac{30}{\pi} \), which must be the required absolute minimal point, which gives \( x = \sqrt[3]{\frac{30}{\pi}} \) as the absolute minimum point.
Finally at the critical point, we have \( 120 = \pi x^2y \), which gives the relation:

\[
120x = \pi x^3y = \pi(\frac{30}{\pi})y = 30y, \text{ so } y = 4x.
\]
So the minimal amount of aluminum is used when the radius of the can is \( \sqrt[3]{\frac{30}{\pi}} = 2.121568836 \) inches and its height is \( 4\sqrt[3]{\frac{30}{\pi}} = 8.486275344 \) inches.
Question 12

For each of the following limits, either determine the limit, or explain why the limit does not exist:

- \( P = \lim_{t \to 2} \left( \frac{t^2 - 6t + 8}{t^4 - 16} \right) \)

\[
P = \lim_{t \to 2} \left( \frac{t^2 - 6t + 8}{t^4 - 16} \right) = \lim_{t \to 2} \left( \frac{(t - 2)(t - 4)}{(t^2 - 4)(t^2 + 4)} \right) = \lim_{t \to 2} \left( \frac{(t - 2)(t - 4)}{(t - 2)(t + 2)(t^2 + 4)} \right)
\]

\[
= \lim_{t \to 2} \left( \frac{(t - 4)}{(t + 2)(t^2 + 4)} \right) = \left( \frac{2 - 4}{2 + 2)((2)^2 + 4) \right) = -\frac{2}{4(8)} = -\frac{1}{16}.
\]

- \( Q = \lim_{t \to 9} \left( \frac{3 - \sqrt{t}}{t^2 - 81} \right) \)

\[
Q = \lim_{t \to 9} \left( \frac{3 - \sqrt{t}}{t^2 - 81} \right) = \lim_{t \to 9} \left( \frac{3 - \sqrt{t}}{(t - 9)(t + 9)} \right) = \lim_{t \to 9} \left( \frac{3 - \sqrt{t}}{(\sqrt{t} - 3)(\sqrt{t} + 3)(t + 9)} \right)
\]

\[
= -\lim_{t \to 9} \left( \frac{1}{(\sqrt{t} + 3)(t + 9)} \right) = -\frac{1}{(\sqrt{9} + 3)(9 + 9)} = -\frac{1}{6(18)} = -\frac{1}{108}.
\]

- \( R = \lim_{t \to \infty} \left( \frac{t^2 - 4 + \sin(t)}{t(t - 8)} \right) \).

As \( t \to \infty \), \( \sin(t) \) oscillates, but is never larger in size than 1, so \( 0 \leq |\sin(t)| \leq \frac{1}{t} \).

Since \( \frac{1}{t} \) goes to zero, by squeeze, so must \( \frac{\sin(t)}{t} \) also. So we get:

\[
R = \lim_{t \to \infty} \left( \frac{t^2 - 4}{t(t - 8)} + \frac{\sin(t)}{t} \right) = \lim_{t \to \infty} \left( \frac{t^2 - 4}{t(t - 8)} \right) + 0
\]

\[
= \lim_{t \to \infty} \left( \frac{t^2(1 - \frac{4}{t^2})}{t^2(1 - \frac{8}{t})} \right) = \lim_{t \to \infty} \left( \frac{(1 - \frac{4}{t^2})}{(1 - \frac{8}{t})} \right) = \frac{1}{1 - 0} = 1.
\]

- \( S = \lim_{t \to 0} \left( \frac{t^2 - 4 + \sin(t)}{t(t - 8)} \right) \).

As \( t \to 0 \), the denominator goes to 0, but the numerator goes to \(-4 \neq 0\).

So the required limit \( S \) does not exist.
Question 13

A triangle $ABC$ in the plane has the following vertices:

$$A = [2, -1], \quad B = [-5, 8], \quad C = [6, -3].$$

- Sketch the triangle $ABC$.
- Find the lengths of the sides of the triangle $ABC$.
  We have:
  - $BC = C - B = [6, -3] - [-5, 8] = [11, -11]$,  
  - $CA = A - C = [2, -1] - [6, -3] = [-4, 2]$,  
  - $AB = B - A = [-5, 8] - [2, -1] = [-7, 9]$,  
  - $a = |BC| = \sqrt{11^2 + (-11)^2} = \sqrt{242} = 11\sqrt{2} = 15.5634918$,  
  - $b = |CA| = \sqrt{(-4)^2 + 2^2} = \sqrt{20} = 2\sqrt{5} = 4.472135954$,  
  - $c = |AB| = \sqrt{(-7)^2 + 9^2} = \sqrt{130} = 11.40175425$.
- Find the angles at the vertices of the triangle $ABC$.
  We have:
  - For the angle at $A$:
    $$\cos(A) = \frac{AB \cdot AC}{|AB| \cdot |AC|} = \frac{[-7, 9] \cdot [4, -2]}{cb} = \frac{-7(4) + 9(-2)}{cb}$$
    $$= \frac{-28 - 18}{\sqrt{130}\sqrt{5}} = \frac{-46}{10\sqrt{26}} = -\frac{23}{5\sqrt{26}}.$$  
    So the angle $A$ is $\arccos\left(-\frac{23}{5\sqrt{26}}\right)$, or $2.695487105$ radians, or $154.4400348$ degrees.
  - For the angle at $B$:
    $$\cos(B) = \frac{BC \cdot BA}{|BC| \cdot |BA|} = \frac{[11, -11] \cdot [7, -9]}{ac} = \frac{11(7) + (-11)(-9)}{ac}$$
    $$= \frac{77 + 99}{11\sqrt{2}\sqrt{130}} = \frac{16}{\sqrt{260}} = \frac{8}{\sqrt{65}}.$$  
    So the angle $B$ is $\arccos\left(\frac{8}{\sqrt{65}}\right)$, or $0.1243549930$ radians, or $7.125016259$ degrees.
- For the angle at $C$:

\[
\cos(C) = \frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\|\overrightarrow{CA}\| \|\overrightarrow{CB}\|} = \frac{[-4, 2],[-11, 11]}{ba} = \frac{-4(-11) + 2(11)}{ba}
\]

\[
= \frac{44 + 22}{11\sqrt{22\sqrt{5}}} = \frac{3}{\sqrt{10}}.
\]

So the angle $C$ is $\arccos\left(\frac{3}{\sqrt{10}}\right)$, or 0.3217505546 radians, or 18.43494883 degrees.

- Find the area of the triangle $ABC$.

If the area is $\Delta$, we have:

\[
\Delta = \frac{1}{2}ab\sin(C),
\]

\[
4\Delta^2 = a^2b^2\sin^2(C) = a^2b^2(1 - \cos^2(C)) = (242)(20)(1 - \frac{9}{10})
\]

\[
= (242)(20)(\frac{1}{10}) = 484,
\]

\[
\Delta^2 = \frac{484}{4} = 121, \quad \Delta = 11.
\]

- Find the equations of the line $L$ through $A$ perpendicular to $BC$ and of the line $M$ through $B$ perpendicular to $AC$; also find the point where the lines $L$ and $M$ meet.

- $BC$ has slope $-\frac{11}{11} = -1$, so $L$ has slope 1.

Then, since $L$ goes through $A = (2, -1)$, the equation of $L$ is

\[
y = (-1) = 1(x - 2), \text{ or } y = x - 3.
\]

- $AC$ has slope $-\frac{2}{4} = -\frac{1}{2}$, so $M$ has slope 2.

Then, since $M$ goes through $B = (-5, 8)$, the equation of $M$ is

\[
y - 8 = 2(x - (-5)), \text{ or } y = 2x + 18.
\]

- The two lines meet where $x - 3 = 2x + 18$, so where $x = -21$ and $y = -24$, so at the point $(-21, -24)$.

Note that $AB$ has slope $\frac{9}{7}$, so the line $N$ perpendicular to $AB$ has slope $\frac{7}{9}$.

If $N$ goes through $C = [6, -3]$, then $N$ has the equation $y - (-3) = \frac{7}{9}(x - 6)$, or $9y + 27 = 7x - 42$, or $9y - 7x = -69$.

Note that the point $(-21, -24)$ also lies on the line $N$, since if $x = -21$ and $y = -24$, we have: $9y - 7x = 9(-24) - 7(-21) = -216 + 147 = -69$. 