Integrated Calculus I Exam 2 Practice Solutions, 10/30/4

Question 1

Find the derivatives of the following functions.

- \(a(t) = \frac{e^{3t}}{\ln^2(t)}\)

  \[a = e^{3t}(\ln(t))^{-2},\]

  \[a' = 3e^{3t}(\ln(t))^{-2} + e^{3t}(-2)(\ln(t))^{-3}\left(\frac{1}{t}\right).\]

- \(b(t) = \arctan(1 + t^2)\)

  \[b' = \frac{1}{1 + u^2}u'|_{u=1+t^2} = \frac{2t}{1 + (1 + t^2)^2}.\]

- \(c(t) = \arcsin(\sqrt{1 - t^2})\)

  \[c' = \frac{1}{\sqrt{1 - u^2}}u'|_{u=\sqrt{1-t^2}} = \frac{1}{\sqrt{1 - (\sqrt{1 - t^2})^2}} + \frac{1}{2}(1 - t^2)^{-\frac{1}{2}}(-2t)\]

  \[= \frac{1}{\sqrt{1 - (1 - t^2)}}\frac{t}{\sqrt{1 - t^2}} = \frac{t}{|t|\sqrt{1 - t^2}}.\]

- \(d(t) = 2^{\ln(t)} \ln(3^{5t^2})\)

  \[d = 2^{\ln(t)}5t^2 \ln(3),\]

  \[d' = 2^{\ln(t)} \ln(2) \left(\frac{1}{t}\right)5t^2 \ln(3) + 2^{\ln(t)}10t \ln(3).\]

  Note that \(2^{\ln(t)} = e^{\ln(2)\ln(t)} = (e^{\ln(t)})^{\ln(2)} = t^{\ln(2)}\) so we have also:

  \[d = t^{\ln(2)}5t^2 \ln(3) = 5 \ln(3)t^{2+\ln(2)},\]

  \[d' = 5 \ln(3)(2 + \ln(2))t^{1+\ln(2)}.\]

- \(e(t) = e^{t^2} \sin^2\left(\frac{1}{t}\right)\)

  \[e' = e^{t^2}(2t) \sin^2\left(\frac{1}{t}\right) + e^{t^2}(2) \sin\left(\frac{1}{t}\right) \cos\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right).\]
Question 2

A curve in the \((x, y)\) plane is given by the formula:
\[
y^4 - 2x^2 y^2 + x^4 + 2x^2 + 2y^2 - 40 = 0.
\]

Find all the points where the slope of the curve is \(\pm 1\), 0 or undefined. What are the dimensions of the smallest rectangular box with sides parallel to the axes, that will enclose the curve? Also plot the curve.

Thinking of \(y\) as an implicit function of \(x\) and differentiating, we get:
\[
4y^3y' - 4xy^2 - 4x^2 yy' + 4x^3 + 4x + 4yy' = 0,
\]
\[
y'(y^3 - x^2 y + y) - xy^2 + x^3 + x = 0.
\]

- The slope is 0 when \(y = 0\) which gives the equation \(0 = -xy^2 + x^3 + x = x(x^2 - y^2 + 1)\), so either when \(x = 0\), or \(y^2 = x^2 + 1\).
  - When \(x = 0\), the equation of the curve gives:
    \[
    0 = y^4 + 2y^2 - 40,
    \]
    \[
    y^2 = \frac{1}{2}(-2 \pm \sqrt{4 + 160}) = -1 \pm \sqrt{41}.
    \]
    Since \(y^2\) is positive we must have:
    \[
    y^2 = -1 + \sqrt{41}, \quad y = \pm \sqrt{\sqrt{41} - 1}.
    \]
    So the curve has horizontal tangents at \((0, \pm \sqrt{\sqrt{41} - 1}) = (0, \pm 2.3245)\).
  - When \(y^2 = x^2 + 1\), the equation of the curve gives:
    \[
    0 = (x^2 + 1)^2 - 2x^2(x^2 + 1) + x^4 + 2x^2 + 2(x^2 + 1) - 40
    \]
    \[
    = x^4 + 2x^2 + 1 - 2x^4 - 2x^2 + x^4 + 2x^2 + 2x^2 + 2 - 40
    \]
    \[
    = 4x^2 - 37
    \]
    \[
    x = \pm \frac{\sqrt{37}}{2}, \quad y^2 = \frac{37}{4} + 1 = \frac{41}{4},
    \]
    \[
    (x, y) = \frac{1}{2}(\sqrt{37}, \pm \sqrt{41}), \frac{1}{2}(-\sqrt{37}, \pm \sqrt{41}).
    \]
    \[
    \]
• Looking at the equation of the curve, we see that it is symmetrical about both axes and also under the interchange of $y$ and $x$: this last interchange interchanges slope 0 points with points where the slope is undefined, so the points with undefined slope are:


• For the third part we first put $y' = 1$, giving the equation:

$0 = y^3 - x^2y + y - xy^2 + x^3 + x = y^3 + x^3 + y + x - (x^2y + xy^2)$

$= (y + x)(y^2 + x^2 - xy) + (y + x)(1) - xy(y + x)$

$= (y + x)(y^2 + x^2 - 2xy + 1)$

$= (y + x)((x - y)^2 + 1)$.

The second term is always positive, so we get $y + x = 0$, or $y = -x$.

Putting this in the equation for the curve, we get:

$x^4 - 2x^2y^2 + x^4 + 2x^2 + 2x^2 - 40 = 0$,

$4x^2 - 40 = 0$,

$(x, y) = \pm (\sqrt{10}, -\sqrt{10})$.

If we reflect a point with slope 1 in the $y$-axis, we get the points with slope $-1$ and vice-versa, so the points of slope $-1$ are $\pm (\sqrt{10}, \sqrt{10})$.

When we sketch the curve we see a rectangular shape, with rounded corners, indented slightly as it crosses the axes.

The points of slope $\pm 1$ give the corners of the shape.

The box in question is the square box bounded by the vertical lines with equation $x = \pm \frac{\sqrt{41}}{2} = \pm 3.2016$ and the horizontal lines $y = \pm \frac{\sqrt{41}}{2} = \pm 3.2016$.  

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Question 3

By using a suitable linear approximation estimate the sine of the angle forty-four degrees.

How can you tell whether your approximation is an over-estimate or an under-estimate?

We put \( f(t) = \sin(t) \), so \( f'(t) = \cos(t) \) and \( f''(t) = -\sin(t) \).

We linearize about the point \( t = \frac{\pi}{4} \), which is the radian measure for forty-five degrees.

We have \( f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \) and \( f'\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \).

Then we have \( f_{lin}(t) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)(t - \frac{\pi}{4}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(t - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(1 + t - \frac{\pi}{4}) \).

For the angle of 44 degrees, we need \( t = \frac{44\pi}{180} \), giving the estimate:

\[
\frac{\sqrt{2}}{2} \left(1 + \frac{44\pi}{180} - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \left(1 - \frac{\pi}{180}\right) = 0.6947654395
\]

the fact that the second derivative is negative, shows that the graph of the function \( \sin(t) \) is concave down near \( t = \frac{\pi}{4} \), so the linearized approximation is an overestimate.

In fact we have \( \sin\left(\frac{44\pi}{180}\right) = 0.6946583704 \), so the estimate is a little more than the true value.

The percentage error is \( 100(0.6947654395 - 0.6946583704)/0.6946583704 = 0.154132 \), or less than two tenths of one percent.
Question 4

Consider the graph of the function \( y = x^4 + 4x^3 - 36x^2 + 135 \).

- Find its critical points.

We have \( y' = 4x^3 + 12x^2 - 72x = 4x(x^2 + 3x - 18) = 4x(x + 6)(x - 3) \), so the critical points are: \( x = 0, x = 3, x = -6 \).
The critical points are then \((0, 135), (3, 0)\) and \((-6, -729)\).

- Find its inflection points.

We have \( y'' = 12x^2 + 24x - 72 = 12(x^2 + 2x - 6) = 0 \), when \( x = \frac{1}{2}(-2 \pm \sqrt{4 + 24}) = -1 \pm \sqrt{7} \). These give inflection points, the points \((\sqrt{7} - 1, -149 + 80\sqrt{7}) = (1.645751311, 62.6601049)\) and \((-\sqrt{7} - 1, -149 - 80\sqrt{7}) = (-3.645751311, -360.6601049)\), since \( y'' \) is positive for large positive or negative \( x \) and yet is negative when \( x = 0 \), so \( y'' \) changes sign through each of these points.

- Find its local maxima and minima.

- When \( x = 0 \), we have \( y'' = -72 \), so \( x = 0 \) is a local maximum.
- When \( x = 3 \), we have \( y'' = 108 \), so \( x = 3 \) is a local minimum.
- When \( x = -6 \), we have \( y'' = 216 \), so \( x = -6 \) is a local minimum.

- Find its absolute maximum and minimum on the interval \([-7, 5]\).

When \( x = 5 \), we have \( y = 360 \). When \( x = -7 \), we have \( y = 2144 \).
So the absolute maximum is 2144 at \( x = -7 \) and the absolute minimum is \(-729 \) at \( x = -6 \).

- Find its intercepts with the \( x \)-axis and with the \( y \)-axis.

The \( y \)-intercept is 135.
We have \( y = x^4 + 4x^3 - 36x^2 + 135 = (x - 3)(x^3 + 7x^2 - 15x - 45) = (x - 3)(x - 3)(x^2 + 10x + 15) = 0 \) when \( x = 3 \), or \( x = -5 \pm \sqrt{10} \). So the \( x \)-intercepts are 3 and \(-5 \pm \sqrt{10}\).

Also plot the graph of the function on the interval \([-7, 5]\).
The graph starts at \((-7, 2144)\), decreases rapidly to \((-7, -729)\) increases to \((0, 135)\), decreases to \((3, 0)\) and then increases to \((5, 360)\).
It is concave up from \( x = -7 \) to \( x = -1 - \sqrt{7} \), then is concave down until \( x = -1 + \sqrt{7} \) and then is concave up.
**Question 5**

Find the linear approximation to the function \(y = \ln(x^2 - 1)\) based at the point with \(x = 3\).

Determine an interval on which the linear approximation differs from the true function by not more than a 2 percent error.

Illustrate your results with suitable plots.

We have at \(x = 3\), \(y = \ln(3^2 - 1) = \ln(9 - 1) = \ln(8) = 3 \ln(2)\).

Also \(y'/|x=3 = \frac{2x}{x^2-1}|x=3 = \frac{2(3)}{3^2-1} = \frac{6}{8} = \frac{3}{4}\).

So the linear approximation based at \(x = 3\) is:

\[y_{lin} = y(3) + y'(3)(x - 3) = 3 \ln(2) + \frac{3}{4}(x - 2).\]

The percentage error is then \(100(y - y_{lin})/y\).

Plotting this, we see that the percentage error is less than 2 percent on the interval \([2.56373, 3.61280]\) approximately.
Question 6

An art object of mass 10 kilos is suspended three meters below a horizontal ceiling in an art gallery by three wires.

Taking the origin directly above the object the object is at the point \( A = [0, 0, -3] \).

Then the wires are attached at points \( B = [4, 0, 0] \), \( C = [0, -4, 0] \) and \( D = [-3, 3, 0] \).

Give a rough sketch of the system of wires.

Find the length of each wire.

Find the forces in the wires.

Sketching the wires, we see that they are symmetrical about the plane with equation \( y + x = 0 \).

This means that the forces in the wires \( AB \) and \( AC \) are the same size.

We have:

\[
AB = B - A = [4, 0, 0] - [0, 0, -3] = [4, 0, 3],
\]

\[
AC = C - A = [0, -4, 0] - [0, 0, -3] = [0, -4, 3],
\]

\[
AD = D - A = [-3, 3, 0] - [0, 0, -3] = [-3, 3, 3].
\]

The wires \( AB \) and \( AC \) both have length \( \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5 \) meters.

The wire \( AD \) has length \( \sqrt{3^2 + 3^2 + 3^2} = \sqrt{27} = 3\sqrt{3} \) meters.

Let the force in the wire \( AD \) be \( yAD \) for some scalar \( y \).

Since \( AB \) and \( AC \) have the same length (5 meters) and the forces in the wires \( AB \) and \( AC \) have the same size, the forces in these wires may be given by \( xAB \) and \( xAC \), for some scalar \( x \).

Then the force balance equation is:

\[
xAB + xAC + yAD + 10[0, 0, -g] = 0,
\]

\[
x([4, 0, 3] + [0, -4, 3]) + y[-3, 3, 3] + [0, 0, -10g] = 0,
\]

\[
x[4, -4, 6] + [-3y, 3y, 3y] + [0, 0, -10g] = 0,
\]

\[
[4x - 3y, -4x + 3y, 6x + 3y - 10g] = 0,
\]

\[
4x - 3y = 0, \quad -4x + 3y = 0, \quad 6x + 3y - 10g = 0,
\]

Adding the first and third of these equations eliminates \( y \) and gives the equation \( 10x - 10g = 0 \), so \( x = g \).
Then the first equation gives $y = \frac{4}{3}x = \frac{4}{3}g$.
So the force in the wire $AB$ is $g[4, 0, 3]$, of size $5g$ Newtons.
So the force in the wire $AC$ is $g[0, -4, 3]$, of size $5g$ Newtons.
Finally the force in the wire $AD$ is $\frac{4}{3}g[-3, 3, 3] = g[-4, 4, 4]$, of size $4g\sqrt{3}$ Newtons.
Note that these three force vectors add to $g[0, 0, 10]$ as they should.

Alternatively, if we did not notice the symmetry in the problem, we write the forces as $xAB$, $zAC$ and $yAD$ for some unknown scalars $x$, $y$ and $z$.
Then the force balance equation is:

$$xAB + zAC + yAD + 10[0, 0, -g] = 0,$$

$$x[4, 0, 3] + z[0, -4, 3] + y[-3, 3, 3] + [0, 0, -10g] = 0,$$

$$[4x, 0, 3x] + [0, -4z, 3z] + [-3y, 3y, 3y] + [0, 0, -10g] = 0,$$

$$[4x - 3y, -4z + 3y, 3x + 3z + 3y - 10g] = 0,$$

$$4x - 3y = 0, -4z + 3y = 0, 3x + 3z + 3y - 10g = 0.$$

From the first equation, we have: $x = \frac{3}{4}y$.
From the second equation, we have: $z = \frac{3}{4}y$.
(So, in particular, it follows that $x = z$).
Inserting these formulas into the third equation, we get:

$$3 \left( \frac{3}{4} \right) y + 3 \left( \frac{3}{4} \right) y + 3y - 10g = 0,$$

$$y \left( \frac{9}{4} + \frac{9}{4} + 3 \right) = 10g,$$

$$y \left( \frac{15}{2} \right) = 10g,$$

$$y = 10g \left( \frac{2}{15} \right) = \frac{4}{3}g,$$

$$x = z = \frac{3}{4}y = \frac{3}{4} \left( \frac{4}{3} \right) g = g.$$

Then we obtain the same results as before for the forces in the wires.
Question 7

Find the minimum distance of the point $A = (-6, \frac{7}{2})$ from the parabola with the equation $y = 2x^2 - 3x + 1$ and find the point $B$ on the parabola closest to the point $A$.

Find the equation of the tangent line $L$ at $B$ and show that the lines $L$ and $AB$ are perpendicular to each other.

Also sketch the parabola and the tangent and normal lines at $B$.

For simplicity we minimize the distance squared, rather than the distance. The distance squared $L = AP^2$ of a point $P = (x, y)$ on the parabola from the point $A = (-6, \frac{7}{2})$ is:

$$L = (x + 6)^2 + (y - \frac{7}{2})^2 = (x + 6)^2 + (2x^2 - 3x - \frac{5}{2})^2.$$

At a local minimum, we have: $\frac{dL}{dx} = 0$, which, using implicit differentiation, gives the equation:

$$0 = \frac{dL}{dx} = 2(x + 6) + 2(y - \frac{7}{2}) \frac{dy}{dx}.$$

Solving this equation gives:

$$m_1 = \frac{dy}{dx} = -\frac{(x + 6)}{(y - \frac{7}{2})}.$$

But $m_1$ is the slope of the tangent line to the parabola at $P$ and the slope $AP$ is $m_2 = \frac{(y - \frac{7}{2})}{(x + 6)}$, so we see that at the critical point, we have $m_1m_2 = -1$, so the slopes are perpendicular, as required.

To solve for the point we substitute $y = 2x^2 - 3x + 1$ and $\frac{dy}{dx} = 4x - 3$ in the critical point equation, giving:

$$0 = \frac{dL}{dx} = 2(x + 6) + 2(y - \frac{7}{2}) \frac{dy}{dx}$$

$$= 2(x + 6) + 2(2x^2 - 3x - \frac{5}{2})(4x - 3).$$

Plotting this, we see a probable root at $x = \frac{3}{2}$.

Multiplying out we get:

$$0 = 2x + 12 + (4x^2 - 6x - 5)(4x - 3)$$
\[ 16x^3 - 36x^2 = 2x + 12 + 16x^3 - 36x^2 - 2x + 15 \]
\[ = 16x^3 - 36x^2 + 27. \]

We factor, expecting a factor of \(2x - 3\):
\[ 16x^3 - 36x^2 + 27 = (2x - 3)(8x^2 - 6x - 9) \]

Finally, we factor the quadratic:
\[ 8x^2 - 6x - 9 = (2x - 3)(4x + 3). \]

So the critical point equation has roots at \(x = \frac{3}{2}\) and \(x = -\frac{3}{4}\) only.

When \(x = \frac{3}{2}\), we have \(y = 2x^2 - 3x + 1 = 2(\frac{9}{4}) - 3\frac{3}{2} + 1 = 1.\)

Then \(L = (\frac{3}{2} + 6)^2 + (1 - \frac{7}{2})^2 = (\frac{15}{2})^2 + (\frac{5}{2})^2 = \frac{225 + 25}{4} = \frac{250}{4}\) and the distance is \(\sqrt{2504} = 5\sqrt{10} = 7.905694150.\)

When \(x = -\frac{3}{4}\), we have \(y = 2x^2 - 3x + 1 = 2(\frac{9}{16}) + 3(\frac{3}{4}) + 1 = \frac{35}{8}.\)

Then \(L = (-\frac{3}{4} + 6)^2 + (\frac{35}{8} - \frac{7}{2})^2 = (\frac{21}{4})^2 + (\frac{7}{8})^2 = \frac{7^2((\frac{3}{2})^2 + (\frac{1}{8})^2) = \frac{7^2}{4} + \frac{1}{64} = \frac{7^2}{64}(37) \) and the distance is \(\frac{7\sqrt{37}}{8} = 5.322417214.\)

On physical grounds we know that there must be an absolute minimum, and since \(L\) is always defined and differentiable, it must be attained at a critical point, so the minimum distance is \(\frac{7\sqrt{37}}{8}\) when \(P = (-\frac{3}{4}, \frac{35}{8}).\)

Then the slope of \(AP\) is \(\frac{\frac{35}{8} - \frac{7}{4} + 6}{-\frac{3}{4} + 6} = \frac{\frac{35}{8} - 28}{-\frac{6}{48}} = \frac{7}{42} = -\frac{1}{6}.\)

Also the slope of the tangent to the parabola at \(P\) is \((4x - 3)|_{x = -\frac{3}{4}} = 4(-\frac{3}{4}) - 3 = -6.\)
Question 8

A particle moving in space has its position $\mathbf{X}$ at time $t$ seconds given by the formula:

$$\mathbf{X} = [8t^2 \sin(t), 8t^2 \cos(t), 6t^2].$$

Obtain formulas for its velocity and speed.

Plot the trajectory of the particle in the first second of its motion.

For which time(s) is the velocity perpendicular to the position and at those times determine the particle’s velocity, speed and position.

Show that for $t \neq 0$, the unit vector in the direction of the particle executes a circle and describe the circle.

The velocity $\mathbf{V}$ is:

$$\mathbf{V} = \frac{d\mathbf{X}}{dt} = [16t \sin(t) + 8t^2 \cos(t), 16t \cos(t) - 8t^2 \sin(t), 12t].$$

Then if $S$ is the speed, we have:

$$S^2 = \mathbf{V} \cdot \mathbf{V} = (16t \sin(t) + 8t^2 \cos(t))^2 + (16t \cos(t) - 8t^2 \sin(t))^2 + 144t^2$$

$$= 256t^2 \sin^2(t) + 256t^4 \sin^2(t) \cos^2(t) + 64t^4 \cos^2(t) + 256t^2 \cos^2(t)$$

$$+ 256t^4 \sin^2(t) \cos(t) + 64t^4 \sin^2(t) + 144t^2$$

$$= 256t^2 + 64t^4 + 144t^2 = 400t^2 + 64t^4 = 16t^2(25 + 4t^2),$$

$$S = 4|t|\sqrt{25 + 4t^2}.$$

The vectors $\mathbf{V}$ and $\mathbf{X}$ are perpendicular when:

$$0 = \mathbf{V} \cdot \mathbf{X} = [16t \sin(t) + 8t^2 \cos(t), 16t \cos(t) - 8t^2 \sin(t), 12t] [8t^2 \sin(t), 8t^2 \cos(t), 6t^2]$$

$$= 8t^2 \sin(t)(16t \sin(t) + 8t^2 \cos(t)) + 8t^2 \cos(t)(16t \cos(t) - 8t^2 \sin(t)) + 72t^3$$

$$= 128t^3 \sin^4(t) + 64t^4 \sin(t) \cos(t) + 128t^3 \cos^2(t) - 64t^4 \sin(t) \cos(t) + 72t^3 = 200t^3.$$ 

So these vectors are perpendicular only when $t = 0$, when the position, velocity and speed all vanish. The length of the vector $\mathbf{X}$ is:

$$|\mathbf{X}| = \sqrt{64t^4 \sin^2(t) + 64t^4 \cos^2(t) + 36t^4} = \sqrt{64t^4 + 36t^4} = 10t^2.$$ 

For $t \neq 0$, the unit vector $\mathbf{N}$ in the direction of $\mathbf{X}$ is:

$$\mathbf{N} = \frac{\mathbf{X}}{|\mathbf{X}|} = \frac{1}{10t^2} [8t^2 \sin(t), 8t^2 \cos(t), 6t^2] = \left[ \frac{4}{5} \sin(t), \frac{4}{5} \cos(t), \frac{3}{5} \right]$$

$$= \frac{4}{5} \sin(t), \cos(t), 0] + [0, 0, \frac{3}{5}].$$

This is the trajectory of a circle centered at $[0, 0, \frac{3}{5}]$.

The circle lies in the horizontal plane $z = \frac{3}{5}$ and has radius $\frac{4}{5}$.  

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**Question 9**

$ABCD$ is a rectangle.

Wire is stretched around the sides $AD$, $CD$ and $BC$. On the fourth side the wire is formed into a semi-circle, from $A$ to $B$, going outside the rectangle, with $AB$ as a diameter.

If the total length of the wire is one meter, what should the dimensions of the rectangle be, in order that the wire enclose the maximum area and what is the maximum possible area?

Let $AB = x = CD$ and $AD = BC = y$ (units are meters). Then the enclosed area $A$ is the sum of the area of a rectangle of sides $x$ and $y$ and the area of a semi-circle of radius $\frac{x}{2}$, so is:

$$A = xy + \frac{1}{2} \left( \pi \left( \frac{x}{2} \right)^2 \right) = xy + \frac{\pi x^2}{8}.$$  

The wire length $P = 1$ meter consists of the sums of the lengths of $AD$, $CD$ and $BC$ which is $2y + x$ and the perimeter of a semicircle of radius $\frac{x}{2}$:

$$P = 2y + x + \frac{1}{2} \left( 2\pi \left( \frac{x}{2} \right) \right) = 2y + \frac{x}{2} (2 + \pi) = 1.$$  

Solving, we get: $y = \frac{1}{2} - \frac{x}{4} (2 + \pi)$. Substituting into the formula for $A$, gives:

$$A = x \left( \frac{1}{2} - \frac{x}{4} (2 + \pi) \right) + \frac{\pi x^2}{8} = \frac{x}{2} - \frac{x^2}{8} (4 + \pi).$$  

The graph of $A$ against $x$ is a downward pointing parabola, so it has exactly one critical point, which is the local and global maximum. We have $A' = \frac{1}{2} - \frac{x}{4} (4 + \pi) = 0$, when $x = \frac{2}{4+\pi}$. So $x = \frac{2}{4+\pi}$ gives the desired maximum area.

When $x = \frac{2}{4+\pi}$, we have $y = \frac{1}{2} - \frac{1}{2(4+\pi)} (2 + \pi) = \frac{4+\pi - 2 - \pi}{2(4+\pi)} = \frac{1}{4+\pi}$.

Then the maximum area is:

$$A = \frac{x}{2} - \frac{x^2}{8} (4 + \pi) = \frac{1}{4+\pi} - \frac{1}{2(4+\pi)} (4+\pi) = \frac{1}{4+\pi} - \frac{1}{2(4+\pi)} = \frac{1}{2(4+\pi)}.$$  

So for the maximum area, we need $CD = \frac{2}{4+\pi} = 0.280050$, $AD = BC = \frac{1}{4+\pi} = 0.140025$ and then the maximum area is $\frac{1}{2(4+\pi)} = 0.070012$ square meters.

Then the rectangle is about 28 centimeters long by 7 meters high, the semicircle has radius 14 cms and the total area is about 700 square centimeters.
Question 10

A piston moving horizontally along the \( x \)-axis is attached by a straight rod of length 30 centimeters, to a point rotating on a circle centered at the origin at a radius of 10 centimeters, which completes 2 revolutions per second. Obtain formulas for the position and velocity of the piston as a function of time and plot their graphs.

The point on the circle rotates at \( 4\pi \) radians per second.

If we take it to start at the point \((0, 10)\) at time \( t = 0 \), then at time \( t \) the point makes an angle of \( 4\pi t \) radians with the positive \( x \)-axis, so it has co-ordinates \( P = (10\cos(4\pi t), 10\sin(4\pi t)) \) (units are centimeters).

Let the piston be at \( Q = (x, 0) \).

Then \( PQ \) is 30 centimeters, which gives the equation:

\[
30^2 = PQ^2 = (x - 10\cos(4\pi t))^2 + (10\sin(4\pi t))^2 = (x - 10\cos(4\pi t))^2 + 100\sin^2(4\pi t),
\]

\[
(x - 10\cos(4\pi t))^2 = 30^2 - 100\sin^2(4\pi t) = 900 - 100\sin^2(4\pi t),
\]

\[
x - 10\cos(4\pi t) = \pm \sqrt{900 - 100\sin^2(4\pi t)} = \pm 10\sqrt{9 - \sin^2(4\pi t)}.
\]

We assume that \( x \) is initially positive, which entails that we take the positive square root, giving the formula for the position:

\[
x = 10\cos(4\pi t) + 10\sqrt{9 - \sin^2(4\pi t)}.
\]

Plotting this gives a fairly straightforward oscillation with maximum amplitude \( x = 40 \) and minimum at \( x = 20 \). The maxima occur at time \( t = 0 \) and then at intervals of one half of a second, so at times \( t + \frac{1}{2} \) with \( n \) an integer. The minima occur at times \( t = \frac{1}{4} \) and then at intervals of one half of a second, so at times \( t = \frac{2m + 1}{4} \), with \( m \) an integer.

The velocity \( v \) is:

\[
v = \frac{dx}{dt} = -40\pi\sin(4\pi t) + \frac{10}{2\sqrt{9 - \sin^2(4\pi t)}}(-2\sin(4\pi t)\cos(4\pi t)(4\pi))
\]

\[
= 40\pi\sin(4\pi t)\left(-1 - \frac{\cos(4\pi t)}{\sqrt{9 - \sin^2(4\pi t)}}\right).
\]

The velocity graph looks a little like a sawtooth, going rapidly from zero to its minimum of \(-132.53 \text{ cms/sec.} \) at time 0.10163 seconds and rising to a maximum of 132.53 cms/sec. at time 0.39837, then falling back to zero again at \( t = \frac{1}{2} \) seconds. Then it repeats every half second.

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