Integrated Calculus I Practice for Exam 1, Solutions 9/29/4

Question 1

Find the derivatives of the following functions. Don’t take the time to simplify your answers:

- \[ a(t) = \frac{t^3 - t + 1}{1 - 2t} \]
  \[ a'(t) = \frac{(3t^2 - 1)(1 - 2t) - (t^3 - t + 1)(-2)}{(1 - 2t)^2} \]

- \[ b(t) = \left(t^2 + \frac{1}{t^2}\right) \left(t - \frac{1}{\sqrt{t}}\right) \]
  \[ b'(t) = \left(2t - \frac{2}{t^3}\right) \left(t - \frac{1}{\sqrt{t}}\right) + \left(t^2 + \frac{1}{t^2}\right) \left(1 + \frac{1}{2}t^{-\frac{3}{2}}\right) \]

Question 2

Find the equations of the tangent and normal lines to the curve \( y = t^2 + \frac{1}{t} \) at the point with \( t = 1 \).
Also sketch the curve and the tangent and normal lines on one graph.

We have:
\[ y' = 2t - t^{-2} = 2t - \frac{1}{t^2}. \]

At \( t = 1 \), we have \( y = 1 + 1 = 2 \) and \( y' = 2 - 1 = 1 \), so the tangent line has slope 1 and passes through the point \((1, 2)\).
By the point slope method, the tangent line has the equation:
\[ y - 2 = 1(t - 1), \quad y = t + 1. \]

The normal line is perpendicular to the tangent line, so has slope \(-1\).
By the point-slope method, the normal line has the equation:
\[ y - 2 = (-1)(t - 1) = 1 - t, \quad y = 3 - t. \]
Question 3

Find the linear approximation to the function \( f(t) = \frac{3}{\sqrt{t}} \), valid near the point with \( t = 9 \).

Sketch the graphs of the function and your linear approximation on the interval \([5, 15]\).

Use your linear approximation to estimate the quantity \( \frac{3}{\sqrt{8}} \) and compare your result with the exact answer.

We have: \( f(9) = \frac{3}{\sqrt{9}} = 1 \); also \( f'(t) = \frac{d}{dt}(3t^{-\frac{1}{2}}) = 3(-\frac{1}{2}t^{-\frac{3}{2}}) \).

So \( f'(9) = -\frac{3}{2} \cdot \frac{1}{\sqrt{9}} = -\frac{3}{2} \cdot \frac{1}{27} = -\frac{1}{18} \).

Then the required linear approximation, \( y_{\text{lin}}(t) \) is:

\[
y_{\text{lin}}(t) = f(9) + f'(9)(t - 9) = 1 - \frac{1}{18}(t - 9) = \frac{1}{18}(27 - t).
\]

The quantity \( \frac{3}{\sqrt{8}} \) is \( y(8) \), so the approximation is: \( y_{\text{lin}}(8) = \frac{1}{18}(27 - 8) = \frac{19}{18} \).

The exact answer is \( \frac{3}{\sqrt{8}} = \sqrt{\frac{9}{4}} = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4} \).

The error is \( \frac{3}{4} - \frac{19}{18} = 0.005104616 \), quite small.

The percentage error is 100\( \frac{\frac{3}{4} - \frac{19}{18}}{\frac{3}{4}} \) = 0.48126786, also quite small.

Question 4

For each of the following limits, either determine the limit, or explain why the limit does not exist:

- \( \lim_{t \to 3} \left( \frac{t^2 - 8t + 15}{t^2 - 9} \right) \)

  We have:

  \[
  \lim_{t \to 3} \left( \frac{t^2 - 8t + 15}{t^2 - 9} \right) = \lim_{t \to 3} \left( \frac{(t - 3)(t - 5)}{(t - 3)(t + 3)} \right) = \lim_{t \to 3} \left( \frac{t - 5}{t + 3} \right) = \frac{3 - 5}{3 + 3} = -\frac{1}{3}.
  \]

- \( \lim_{t \to 4} \left( \frac{2 - \sqrt{t}}{t^2 - 6t + 8} \right) \)

  We have:

  \[
  \lim_{t \to 4} \left( \frac{2 - \sqrt{t}}{t^2 - 6t + 8} \right) = \lim_{t \to 4} \left( \frac{(2 - \sqrt{t})(2 + \sqrt{t})}{(t - 2)(t - 4)(2 + \sqrt{t})} \right) = \lim_{t \to 4} \left( \frac{4 - t}{(t - 2)(t - 4)(2 + \sqrt{t})} \right)
  \]

  \[
  = \lim_{t \to 4} \left( \frac{-1}{(t - 2)(2 + \sqrt{t})} \right) = -\frac{1}{(4 - 2)(2 + \sqrt{4})} = -\frac{1}{8}.
  \]
Question 5

Given that two functions $f(t)$ and $g(t)$ obey the relations $f'(t) = g(t)$ and $g'(t) = -f(t)$, determine the following derivatives:

- \[ \frac{d}{dt} \left( (f(t))^2 + (g(t))^2 \right). \]
  
  We have:
  \[
  (f^2 + g^2)' = 2ff' + 2gg' = 2f(g) + 2g(-f) = 2fg - 2fg = 0.
  \]

- \[ \frac{d}{dt} (f(t)g(t)). \]
  
  We have:
  \[
  (fg)' = fg' + f'g = f(-f) + g(g) = g^2 - f^2.
  \]

- \[ \frac{d}{dt} \left( \frac{f(t)}{g(t)} \right). \]
  
  We have:
  \[
  \left( \frac{f}{g} \right)' = \frac{f'g - fg'}{g^2} = \frac{g(g) - f(-f)}{g^2} = \frac{f^2 + g^2}{g^2}.
  \]

Given also that $f(0) = 2$ and $g(0) = 0$, find the equation of the tangent line to the curve $y = f(t) + g(t)$ at $t = 0$.

When $t = 0$, we have $y = f(0) + g(0) = 2 + 0 = 2$.
So the required tangent line goes through the point $(0, 2)$.
The slope of the curve is $y' = f'(t) + g'(t) = g(t) - f(t)$.
Putting $t = 0$ gives $y' = g(0) - f(0) = 0 - 2 = -2$.
So the required tangent line has slope $-2$.
So the required tangent line has the equation:
\[
  y - 2 = (-2)(t - 0),
  \]
\[
  y = 2 - 2t.
  \]
Question 6
A particle in the plane has position \( X(t) = [x(t), y(t)] \) at time \( t \) seconds given as follows (units are metric):
\[
X(t) = [x(t), y(t)] = \left[ \frac{2t}{1 + t^2}, \frac{1 - t^2}{1 + t^2} \right].
\]

- Where is the particle at time \( t = \frac{1}{2} \) second?

Putting \( t = \frac{1}{2} \) we find that the particle is at:
\[
X\left(\frac{1}{2}\right) = \left[ \frac{2\left(\frac{1}{2}\right)}{1 + \left(\frac{1}{2}\right)^2}, \frac{1 - \left(\frac{1}{2}\right)^2}{1 + \left(\frac{1}{2}\right)^2} \right] = \left[ \frac{1}{\left(\frac{5}{4}\right)}, \frac{\left(\frac{3}{4}\right)}{\left(\frac{5}{4}\right)} \right] = \left[ \frac{4}{5}, \frac{3}{5} \right].
\]

- Find the particle’s velocity \( V(t) = \frac{dX}{dt} = \left[ \frac{dx}{dt}, \frac{dy}{dt} \right] \) as a function of the time.

We have using the division rule:
\[
V = \left[ \frac{2(1 + t^2) - 2t(2t)}{(1 + t^2)^2}, \frac{-2t(1 + t^2) - (1 - t^2)(2t)}{(1 + t^2)^2} \right] = \frac{2}{(1 + t^2)^2} [1 - t^2, -2t].
\]

Putting \( t = \frac{1}{2} \), we get:
\[
V\left(\frac{1}{2}\right) = \frac{2}{\left(\frac{5}{4}\right)^2} \left[ \frac{3}{4}, -1 \right] = \frac{32}{25} \left[ \frac{3}{4}, -1 \right] = \frac{8}{25} [3, -4].
\]

- Does a time \( t \) exist at which the velocity and position vectors are parallel, perpendicular or equal? Explain your answers.

\[
X.V = \frac{1}{(1 + t^2)^3} [2t, 1 - t^2]. [1 - t^2, -2t] = \frac{1}{(1 + t^2)^3} (2t(1 - t^2) + (1 - t^2)(-2t)) = 0.
\]
So the velocity and position vectors are always perpendicular. Then they can be parallel or equal only if one or the other is the zero vector. But the two components of each vector are proportional to the quantities \( 2t \) and \( 1 - t^2 \), which never both vanish, so neither vector is ever the zero vector, so the vectors are never equal and never proportional.

- What is the speed of the particle at time \( t = \frac{1}{2} \)?

The speed of the particle in meters per second is the size of the velocity vector, so at time \( t = \frac{1}{2} \) is:
\[
|V| = \frac{8}{25} \sqrt{3^2 + (-4)^2} = \frac{8}{25} \sqrt{25} = \frac{8}{5}.
\]
Question 7

A ten kilogram mass is hanging in a room at point $C = [4, 4]$. It is attached by straight strings to points $A = [0, 8]$ and $B = [12, 8]$ on the ceiling above the mass. Find the forces in the strings $AC$ and $BC$, assuming that the strings have negligible mass.

We have $F = xCA$ and $G = yCB$ as the forces in the strings $CA$ and $CB$, respectively.

Then we solve the force balance equation:

\[ F + G + [0, -10g] = [0, 0], \]
\[ xCA + yCB + [0, -10g] = [0, 0], \]
\[ x(A - C) + y(B - C) + [0, -10g] = [0, 0], \]
\[ x[-4, 4] + y[8, 4] + [0, -10g] = [0, 0], \]
\[ [-4x + 8y, 4x + 4y - 10g] = [0, 0], \]
\[ -4x + 8y = 0, \quad 4x + 4y - 10g = 0, \]
\[ x = 2y, \quad 8y + 4y = 10g, \]
\[ y = \frac{5g}{6}, \quad x = \frac{5g}{3}. \]

So the force in the string $AC$ is:

\[ F = \frac{5g}{3} [-4, 4] = \frac{20g}{3} [-1, 1]. \]

This force has size in Newtons:

\[ |F| = \frac{20g}{3} \sqrt{(-1)^2 + 1^2} = \frac{20g\sqrt{2}}{3} = 92.39528605. \]

Finally the force in the string $BC$ is:

\[ G = \frac{5g}{6} [8, 4] = \frac{10g}{3} [2, 1]. \]

This force has size in Newtons:

\[ |G| = \frac{10g}{3} \sqrt{2^2 + 1^2} = \frac{10g\sqrt{5}}{3} = 73.04488726. \]
Question 8

A child drags her 10 kilogram sled 30 meters smoothly up a snow covered slope.
The slope makes an angle of 12 degrees with the horizontal.
The child applies a constant force of size 150 Newtons to the sled, at an angle of 15 degrees to the slope.

- What is the force due to gravity on the sled (the acceleration due to gravity is 9.8 meters per second per second)?

With the $x$-axis horizontal and the $y$-axis vertical, with $y$ positive in the upward direction, the gravitational force $G$ on the sled is $G = [0, -10g] = [0, -98]$, the units being Newton’s.

- What is the resultant force of gravity and the child, acting on the sled? How large is the force?

The child’s force $C$ is of size 150 Newtons at a total angle of 27 degrees or $\frac{27\pi}{180} = \frac{3\pi}{20}$ radians to the horizontal, so is given by the formula: $C = [150 \cos(\frac{3\pi}{20}), 150 \sin(\frac{3\pi}{20})]$. The required resultant force $F$ is then: $F = C + G = [150 \cos(\frac{3\pi}{20}), 150 \sin(\frac{3\pi}{20}) - 98] = [133.6509786, -29.90142503]$.

The size of this force in Newtons is then:

$$|F| = \sqrt{(150 \cos(\frac{3\pi}{20}))^2 + (150 \sin(\frac{3\pi}{20}) - 98)^2}$$

$$= \sqrt{(150)^2((\cos(\frac{3\pi}{20}))^2 + (\sin(\frac{3\pi}{20}))^2) + 98^2 - 2(98)\sin(\frac{3\pi}{20})}$$

$$= \sqrt{(150)^2 + 98^2 - 2(98)(150)\sin(\frac{3\pi}{20})} = \sqrt{32104 - 29400\sin(\frac{3\pi}{20})} = 136.9550266.$$  

- How much work is done by the child on the sled in dragging the sled the 30 meters?

The required work done, $W$ Joules is the product of the distance moved 30 meters and the component of the child’s force in the direction of the motion, which is $150 \cos(\frac{15\pi}{180}) = 150 \cos(\frac{\pi}{12})$ Newtons, which gives:

$$W = 30(150 \cos(\frac{\pi}{12})) = 4500(\sqrt{2} + \sqrt{6}) = 1125(\sqrt{2} + \sqrt{6}) = 4346.666218.$$
Question 9

Let \( f(x) = x + \frac{4}{x} \), defined for any \( x \neq 0 \).

- From first principles, compute the following limits:
  \[ L_1 = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}, \quad L_2 = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}. \]

We have \( f(1) = 1 + 4 = 5 \) and \( f(2) = 2 + 2 = 4 \), so the limits are:

\[
L_1 = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x + \frac{4}{x} - 5}{x - 1} = \lim_{x \to 1} \frac{x^2 + 4 - 5x}{x(x - 1)}
= \lim_{x \to 1} \frac{(x - 1)(x - 4)}{x(x - 1)} = \lim_{x \to 1} \frac{x - 4}{x} = \frac{1 - 4}{1} = -3
\]

\[
L_2 = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{x + \frac{4}{x} - 4}{x - 2} = \lim_{x \to 2} \frac{x^2 + 4 - 4x}{x(x - 2)}
= \lim_{x \to 2} \frac{(x - 2)(x - 2)}{x(x - 2)} = \lim_{x \to 2} \frac{x - 2}{x} = 0.
\]

- Give the geometrical interpretation of the limits \( L_1 \) and \( L_2 \).
  These are the tangent slopes at \( x = 1 \) and \( x = 2 \), to the curve \( y = f(x) \), respectively.

- Find the equations of the tangent and normal lines to the curve \( y = f(x) \) at \( x = 1 \) and at \( x = 2 \).
  The tangent line at \((1,5)\) is: \( y - 5 = -3(x - 1) \), or \( y = -3x + 8 \).
  The normal line at \((1,5)\) is: \( -y - 5 = \frac{1}{3}(x - 1) \), or \( x - 3y = -14 \).
  The tangent line at \((2,4)\) is horizontal, so is: \( y = 4 \).
  The normal line at \((2,4)\) is vertical, so is: \( x = 2 \).

- Find the meeting point of the two tangent lines and the angle between these two lines.
  The line \( x = 2 \) meets the line \( y = -3x + 8 \) at \((2,2)\).
  The angle between the lines is \( \frac{\pi}{2} + \arctan(3) = 2.819842099 \) radians, or 161.5650511 degrees.
Question 10

A triangle $ABC$ in the plane has the vertices:

\[ A = [2, -1], \quad B = [4, 3], \quad C = [3, -2]. \]

- Find the centroid of the triangle $ABC$.
  
  The centroid $G$ is at the vector average of the vertex vectors, so is at:
  \[ G = \frac{1}{3}(A + B + C) = \frac{1}{3}([2, -1] + [4, 3] + [3, -2]) = \frac{1}{3}([9, 0]) = [3, 0]. \]

- Find the angles of the triangle $ABC$.
  
  \[
  BC = C - B = [3, -2] - [4, 3] = [-1, -5], \quad a = |BC| = \sqrt{(-1)^2 + (-5)^2} = \sqrt{26},
  \]
  \[
  CA = A - C = [2, -1] - [3, -2] = [-1, 1], \quad b = |CA| = \sqrt{(-1)^2 + 1^2} = \sqrt{2},
  \]
  \[
  AB = B - A = [4, 3] - [2, -1] = [2, 4], \quad c = |AB| = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5},
  \]
  \[
  \cos(A) = \frac{AB \cdot AC}{|AB||AC|} = \frac{[2,4],[1,-1]}{2\sqrt{10}} = \frac{2 - 4}{2\sqrt{10}} = -1/\sqrt{10},
  \]
  \[
  \cos(B) = \frac{BA \cdot BC}{|BA||BC|} = \frac{[-2,-4],[-1,-5]}{2\sqrt{130}} = \frac{2 + 20}{2\sqrt{130}} = \frac{11}{\sqrt{130}},
  \]
  \[
  \cos(C) = \frac{CA \cdot CB}{|CA||CB|} = \frac{[-1,1],[1,5]}{\sqrt{52}} = \frac{-1 + 5}{\sqrt{52}} = \frac{2}{\sqrt{13}}.
  \]

So the angles are as follows:

$A$ is arccos($-1/\sqrt{10}$) = 1.892546882 radians, or 108.4349488 degrees.

$B$ is arccos($11/\sqrt{130}$) = 0.2662520493 radians, or 15.25511871 degrees.

$C$ is arccos($2/\sqrt{13}$) = 0.9827937235 radians, or 56.30993248 degrees.

Note that the sum of these angles is about $\pi$ radians or 180 degrees, as expected.

- Find the area of the triangle $ABC$.

  If $\Delta$ is the area, we have: $\Delta = \frac{1}{2}bc\sin(A)$, which gives:
  \[
  \Delta^2 = \frac{1}{4}b^2c^2\sin^2(A) = \frac{1}{4}b^2c^2(1 - \cos^2(A)) = \left(\frac{1}{4}\right)2(20)(1 - \frac{1}{10}) = 9.
  \]
  So $\Delta = 3$.

- If $ABCD$ is a parallelogram, with $AB$ parallel to $CD$, where is the point $D$?

  We need $\overrightarrow{AB} = \overrightarrow{CD}$, which gives the equation:
  \[
  [2, 4] = \overrightarrow{CD} = \overrightarrow{D} - \overrightarrow{C} = \overrightarrow{D} - [3, -2]. \quad \text{So } \overrightarrow{D} = [2, 4] + [3, -2] = [5, 2].
  \]