A Weak Dynamic Programming Principle for Zero-Sum Stochastic Differential Games with Unbounded Controls

Erhan Bayraktar∗†, Song Yao‡

Abstract

We analyze a zero-sum stochastic differential game between two competing players who can choose unbounded controls. The payoffs of the game are defined through backward stochastic differential equations. We prove that each player’s priority value satisfies a weak dynamic programming principle and thus solves the associated fully non-linear partial differential equation in the viscosity sense.

Keywords: Zero-sum stochastic differential games, Elliott-Kalton strategies, weak dynamic programming principle, backward stochastic differential equations, viscosity solutions, fully non-linear PDEs.

Contents

1 Introduction 1

1.1 Notation and Preliminaries . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3
1.2 Backward Stochastic Differential Equations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4

2 Stochastic Differential Games with Super-square-integrable Controls 5

2.1 Game Setting: A Controlled SDE−BSDE System . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5
2.2 Definition of Value Functions and a Weak Dynamic Programming Principle . . . . . . . . . . . . . 7

3 Viscosity Solutions of Related Fully Non-linear PDEs 9

4 Proofs 10

1 Introduction

In this paper we extend the study of Buckdahn and Li [10] on a zero-sum stochastic differential game (SDG), whose payoffs are generated by backward stochastic differential equations (BSDEs), to the case of super-square-integrable controls (see Remark 2.1).

Since its initiation by Fleming and Souganidis [15], the SDG theory has grown rapidly in many aspects (see e.g. the references in [10, 9]). Among these developments, Hamadène et al. [17, 18, 13] introduced a (decoupled) SDE−BSDE system, with controls only in the drift coefficients, to generate the payoffs in their studies of saddle point problems of SDGs. (For the evolution and applications of the BSDE theory, see Pardoux and Peng [24], El Karoui et al. [14] and the references therein.) Later on, [10] as well as its sequels [12, 11, 9] generalized the SDE−BSDE framework so that the two competing controllers can also influence the diffusion coefficient of the state dynamics. Unlike [15], [10] used a uniform canonical space Ω = {ω ∈ C([0,T]; Rd) : ω(0) = 0} so that admissible control processes can also depend on the information occurring before the start of the game. Such a setting allows the authors of [10] get around a relatively complicated approximation argument of [13] which was due

∗Department of Mathematics, University of Michigan, Ann Arbor, MI 48109; email: erhan@umich.edu.
†E. Bayraktar is supported in part by the National Science Foundation under applied mathematics research grants and a Career grant, DMS-0906257, DMS-1118673, and DMS-0955463, respectively, and in part by the Susan M. Smith Professorship.
‡Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260; email: songyao@pitt.edu.
to a measurability issue (see Remark 2.6), and allows them to adopt the notion of stochastic backward semigroups and a BSDE method, developed in [25, 27], to obtain results similar to [15]: the lower and upper values of the SDG satisfy a dynamic programming principle and solve the associated Hamilton-Jacobi-Bellman-Isaacs equations in the viscosity sense. However, [10, 15] as well as some latest advances to the SDG theory (e.g. Bouchard et al. [5] on stochastic target games, Peng and Xu [26] on SDGs in form of a generalized BSDE with random default time) still assume the compactness of control spaces while Pham and Zhang [29] on weak formulation of SDGs assumes the boundedness of coefficients in control variables. In this paper, we are going to address these particular issues.

In the present paper, since two players take super-square-integrable admissible controls over two separable metric spaces $U$ and $V$, those approximation methods of [15] and [10] in proving the dynamic programming principle are no longer effective. Instead, We derive a weak form of dynamic programming principle in spirit of Bouchard and Touzi [6] and use it to show that each player’s priority value solves the corresponding fully non-linear PDE in the viscosity sense. Vitoria [30] has tried to extend the SDG for unbounded controls by proving a weak dynamic programming principle. However, it still assumed that the control space of the player with priority is compact, see Theorem 75 therein.

Square-integrable controls were initially considered by Krylov [21] Chapter 6), however, for cooperative games (i.e. the so called sup sup case). Browne [8] studied a specific zero-sum investment game between two small investors who control the game via their square-integrable portfolios. Since the PDEs in this case have smooth solutions, the problem can be solved by a verification theorem instead of the dynamic programming principle. It is also worth mentioning that inspired by the “tug-of-war” (a discrete-time random turn game, see e.g. [28] and [22]), Atar and Budhiraja [1] studied a zero-sum stochastic differential game with $U = V = \{ x \in \mathbb{R}^n : |x| = 1 \} \times [0, \infty)$ played until the state process exits a given domain. As in Chapter 6 of [21], the authors approximated such a game with unbounded controls by a sequence of games with bounded controls which satisfy a dynamic programming principle. They showed the equicontinuity of the approximating sequence and thus proved that the value function of the game is a unique viscosity solution to the inhomogenous infinity Laplace equation. We do not rely on this approximation scheme but directly prove a weak dynamic programming principle for the game with super-square-integrable controls.

Following the probabilistic setting of [10] (see Remark 2.6), our paper takes the canonical space $\Omega = \{ \omega \in \mathbb{C}([0, T]; \mathbb{R}^d) : \omega(0) = 0 \}$, whose coordinator process $B$ is a Brownian motion under the Wiener measure $P$. When the game starts from time $t \in [0, T]$, under the super-square-integrable controls $\mu \in U_t$ and $\nu \in V_t$ selected by player I and II respectively, the state process $X^t,\mu,\nu$ starting from a random initial state $\xi$ will then evolve according to a stochastic differential equation (SDE):

$$X_s = \xi + \int_t^s b(r, X_r, \mu_r, \nu_r) \, dr + \int_t^s \sigma(r, X_r, \mu_r, \nu_r) \, dB_r, \quad s \in [t, T],$$  

where the drift $b$ and the diffusion $\sigma$ are Lipschitz continuous in $x$ and have linear growth in $(u, v)$. The payoff player I will receive from player II is determined by the first component of the unique solution $(Y^{t,\xi,\mu,\nu}, Z^{t,\xi,\mu,\nu})$ to the following BSDE:

$$Y_s = g(X^t,\mu,\nu) + \int_s^T f(r, X_r, \mu_r, \nu_r, Y_r, Z_r, \mu_r, \nu_r) \, dr - \int_s^T Z_r \, dB_r, \quad s \in [t, T].$$  

Here the generator $f$ is Lipschitz continuous in $(y, z)$ and also has linear growth in $(u, v)$. When $g$ and $f$ are $2/p$–Hölder continuous in $x$ for some $p \in (1, 2]$, $Y^{t,\xi,\mu,\nu}$ is $p$–integrable. As we see from (1.1) and (1.2) that the controls $\mu, \nu$ influence the game in two aspects: either affect (1.2) via the state process $X^{t,\xi,\mu,\nu}$ or appear directly in the generator $f$ of (1.2) as parameters. In particular, if $f$ is independent of $(y, z)$, $Y$ is in form of the conditional linear expectation of the terminal reward $g(X^T,\mu,\nu)$ plus the cumulative reward $\int_s^T f(r, X^t,\mu,\nu) \, dr$ (cf. [15]).

When the player (e.g. Player I) with the priority chooses firstly a super-square-integrable control (e.g. $\mu \in U_t$), its opponent (e.g. Player II) will select its reacting control via a non-anticipative mapping $\beta_t : U_t \to V_t$, called Elliott-Kalton strategy, due to some technical subtleties as demonstrated in [15]. In particular, using Elliott-Kalton strategies is essential in proving the dynamic programming principle. This phenomenon already appears in the controller-stopper games, i.e. when one of the players is endowed with the right of stopping the game instead of
using a control; see [2], which shows that if the stopper acts second it is necessary that the stopper uses non-anticipative strategies in order to prove a dynamic programming principle. This type of phenomenon does not appear (or it is implicitly satisfied) if the controllers only control the drift, see e.g. [3] and the references therein, or when there are two stoppers (the so-called Dynkin games), see e.g. [4] and the references therein.

By \( w_1(t, x) \overset{\Delta}{=} \text{essinf } \text{esssup} Y^t_{\mu, \beta} \left( \tau_{\beta, \mu}, \phi \left( \tau_{\beta, \mu}, X_{\tau_{\beta, \mu}, \mu} \right) \right) \) we will represents Player I’s priority value of the game starting from time \( t \) and state \( x \), where \( \mathcal{B}_t \) collects all admissible strategies for Player II. Switching the priority defines Player II’s priority value \( w_2(t, x) \).

Although our setting makes the payoffs \( Y^t_{\xi, \mu, \nu} \) random variables, we can show like [10] that \( w_1(t, x) \) and \( w_2(t, x) \) are invariant under Girsanov transformation via functions of the Cameron-Martin space and are thus deterministic, see Lemma 2.2. To assure values \( w_1(t, x) \) and \( w_2(t, x) \) are finite, we assume that each player has some control neutralizer for coefficients \((b, \sigma, f)\) (such an assumption holds for additive controls, see Remark 2.2), and we impose a growth condition on strategies. These two technical requirements also plays an important role in proving our weak dynamic programming principle. When \( U \) and \( V \) are compact, the control neutralizers become futile and the growth condition holds automatically for strategies. Thus our problem degenerates to [10]’s case, see Remark 2.3.

Although value functions \( w_1(t, x), w_2(t, x) \) are still \( 2/p \)-Hölder continuous in \( x \) (see Proposition 2.3), they may not be continuous in \( t \). Hence we can not follow [10]’s approach to get a strong form of dynamic programming principle for \( w_1 \) and \( w_2 \). Instead, we prove a weak dynamic programming principle, say for \( w_1 \):

\[
\text{essinf } \text{esssup}_{\beta \in \mathcal{B}_t} \text{esssup}_{\mu \in U_t} Y^t_{\mu, \beta}(\tau_{\beta, \mu}, \phi(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}, \mu})) \leq w_1(t, x) \leq \text{essinf } \text{esssup}_{\beta \in \mathcal{B}_t} \text{esssup}_{\mu \in U_t} Y^t_{\mu, \beta}(\tau_{\beta, \mu}, \phi(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}, \mu}))
\]

for any two continuous functions \( \phi \leq w_1 \leq \tilde{\phi} \). Here \( \tau_{\beta, \mu} \) denotes the first existing time of state process \( X_{\tau_{\beta, \mu}, \mu} \) from the given open ball \( O_\delta(t, x) \).

To prove the weak dynamic programming principle, we first approximate \( w_1(t, x) = \text{essinf } I(t, x, \beta) \) from above \( \beta \in \mathcal{B}_t \), and \( I(t, x, \beta) \overset{\Delta}{=} \text{essinf } Y^t_{\mu, \beta} \) from below in a probabilistic sense (see Lemma 4.2) so that we can construct \( \varepsilon \)-optimal controls/strategies by a pasting technique similar to the one used in [6] and [30]. Then we make a series of estimates and eventually obtain the weak dynamic programming principle by using a stochastic backward semigroup property (2.10), the continuity dependence of payoff process \( Y^t_{\xi, \mu, \nu} \) on \( \xi \) (see Lemma 2.3) as well as the control-neutralizer assumption and the growth condition on strategies.

Next, one can deduce from the weak dynamic programming principle and the separability of control space \( U \), \( V \) that the value functions \( w_1 \) and \( w_2 \) are (discontinuous) viscosity solutions of the corresponding fully non-linear PDEs, see Theorem 3.1.

The rest of the paper is organized as follows: After listing the notations to use, we recall some basic properties of BSDEs in Section 1. In Section 2, we set up the zero-sum stochastic differential games based on BSDEs and present a weak dynamic programming principle for priority values of both players defined via Elliott-Kalton strategies. With help of the weak dynamic programming principle, we show in Section 3 that the priority values are (discontinuous) viscosity solutions of the corresponding fully non-linear PDEs. The proofs of our results are deferred to Section 4.

### 1.1 Notation and Preliminaries

Let \((M, \rho_M)\) be a generic metric space and let \( \mathcal{B}(M) \) be the Borel \( \sigma \)-field on \( M \). For any \( x \in M \) and \( \delta > 0 \), \( O_\delta(x) \overset{\Delta}{=} \{ x' \in M : \rho_M(x, x') < \delta \} \) and \( \overline{O}_\delta(x) \overset{\Delta}{=} \{ x' \in M : \rho_M(x, x') \leq \delta \} \) respectively denote the open and closed ball centered at \( x \) with radius \( \delta \). For any function \( \phi : M \to \mathbb{R} \), we define

\[
\lim_{x' \to x} \phi(x') \overset{\Delta}{=} \lim_{n \to \infty} \inf_{x' \in O_{1/n}(x)} \phi(x') \quad \text{and} \quad \lim_{x' \to x} \phi(x') \overset{\Delta}{=} \lim_{n \to \infty} \sup_{x' \in \overline{O}_{1/n}(x)} \phi(x'), \quad \forall x \in M.
\]

Fix \( d \in \mathbb{N} \) and a time horizon \( T \in (0, \infty) \). We consider the canonical space \( \Omega \overset{\Delta}{=} \{ \omega \in C([0, T]; \mathbb{R}^d) : \omega(0) = 0 \} \) equipped with Wiener measure \( P \), under which the canonical process \( B \) is a \( d \)-dimensional Brownian motion.
Let $\mathbf{F} = \{ \mathcal{F}_t \}_{t \in [0, T]}$ be the filtration generated by $B$ and augmented by all $P$–null sets. We denote by $\mathcal{P}$ the $\mathbf{F}$–progressively measurable $\sigma$–field of $[0, T] \times \Omega$.

Given $t \in [0, T]$, let $\mathcal{S}_{t,T}$ collect all $\mathbf{F}$–stopping times $\tau$ with $t \leq \tau \leq T$, $P$–a.s. For any $\tau \in \mathcal{S}_{t,T}$ and $A \in \mathcal{F}_\tau$, we define $[t, \tau]_A \triangleq \{(r, \omega) \in [t, T] \times A : r < \tau(\omega)\}$ and $[\tau, T]_A \triangleq \{(r, \omega) \in [t, T] \times A : r \geq \tau(\omega)\}$ for any $A \in \mathcal{F}_\tau$.

In particular, $[t, \tau]_\Omega$ and $[\tau, T]_\Omega$ are the stochastic intervals.

Let $\mathbb{E}$ be a generic Euclidean space. For any $p \in [1, \infty)$ and $t \in [0, T]$, we introduce some spaces of functions:

1. For any $p \in [1, \infty)$ and $[t, \tau]_\Omega$ and $[\tau, T]_\Omega$ are the stochastic intervals. For any $p \in [1, \infty)$ and $t \in [0, T]$, we introduce some spaces of functions:

2. $C_p^p([t, T], \mathbb{E})$ denotes the space of all $\mathbb{E}$–valued, $\mathbf{F}$–adapted processes $\{X_s\}_{s \in [t, T]}$ with $P$–a.s. continuous paths such that $\|X\|_{C_p^p([t, T], \mathbb{E})} \triangleq \left\{E\left[\sup_{s \in [t, T]} |X_s|^p\right]\right\}^{1/p} < \infty$.

3. $H_{p,\text{loc}}^p([t, T], \mathbb{E})$ denotes the space of all $\mathbb{E}$–valued, $\mathbf{F}$–progressively measurable processes $\{X_s\}_{s \in [t, T]}$ such that $\int_t^T |X_s|^p \, ds < \infty$, $P$–a.s. For any $p \in [1, \infty)$, $H_{p,\text{loc}}^p([t, T], \mathbb{E})$ denotes the space of all $\mathbb{E}$–valued, $\mathbf{F}$–progressively measurable processes $\{X_s\}_{s \in [t, T]}$ with $\|X\|_{H_{p,\text{loc}}^p([t, T], \mathbb{E})} \triangleq \left\{E\left[\left(\int_t^T |X_s|^p \, ds\right)^{\frac{p}{p'}}\right]\right\}^{1/p} < \infty$.

4. We also set $\mathbb{G}_p^p([t, T]) \triangleq C_p^p([t, T], \mathbb{R}) \times H_{p,\text{loc}}^p([t, T], \mathbb{R}^d)$.

If $\mathbb{E} = \mathbb{R}$, we will drop it from the above notations. Moreover, we will use the convention $\inf \emptyset = \infty$.

### 1.2 Backward Stochastic Differential Equations

Given $t \in [0, T]$, a $t$–parameter set $(\eta, f)$ consists of a random variable $\eta \in L^0(\mathcal{F}_t)$ and a function $f : [t, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ that is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) / \mathcal{B}(\mathbb{R})$–measurable. In particular, $(\eta, f)$ is called a $(t, p)$–parameter set for some $p \in [1, \infty)$ if $\eta \in L^p(\mathcal{F}_t)$.

**Definition 1.1.** Given a $t$–parameter set $(\eta, f)$ for some $t \in [0, T]$, a pair $(Y, Z) \in C_p^0([t, T]) \times H_{p,\text{loc}}^p([t, T], \mathbb{R}^d)$ is called a solution of the backward stochastic differential equation on the probability space $(\Omega, \mathcal{F}_t, P)$ over period $[t, T]$ with terminal condition $\eta$ and generator $f$ (BSDE$(t, \eta, f)$ for short) if it holds $P$–a.s. that

$$Y_s = \eta + \int_s^T f(r, Y_r, Z_r) \, dr - \int_s^T Z_r \, dB_r, \quad s \in [t, T].$$

Analogous to Theorem 4.2 of [2], we have the following well-posedness result of BSDE (1.3).

**Proposition 1.1.** Given $t \in [0, T]$ and $p \in [1, \infty)$, let $(\eta, f)$ be a $(t, p)$–parameter set such that $f$ is Lipschitz continuous in $(y, z)$: i.e. for some $\gamma > 0$, it holds for $ds \times dP$–a.s. $(s, \omega) \in [t, T] \times \Omega$ that

$$|f(s, \omega, y, z) - f(s, \omega, y', z')| \leq \gamma(|y - y'| + |z - z'|), \quad \forall y, y' \in \mathbb{R}, \quad \forall z, z' \in \mathbb{R}^d.$$ If $E\left[\left(\int_t^T |f(s, 0, 0)| \, ds\right)^p\right] < \infty$, BSDE (1.3) admits a unique solution $(Y, Z) \in \mathbb{G}_p^p([t, T])$ such that satisfies

$$E\left[\sup_{s \in [t, T]} |Y_s|^p\right]_{\mathcal{F}_t} \leq C(T, p, \gamma)E\left[|\eta|^p + \left(\int_t^T |f(s, 0, 0)| \, ds\right)^p\right]_{\mathcal{F}_t}, \quad P$–a.s. (1.4)

Also, we have the following priori estimate and comparison for BSDE (1.3).

**Proposition 1.2.** Given $t \in [0, T]$ and $p \in [1, \infty)$, let $(\eta_i, f_i), i = 1, 2$ be two $(t, p)$–parameter sets such that $f_1$ is Lipschitz continuous in $(y, z)$, and let $(Y^{1, i}, Z^{1, i}) \in \mathbb{G}_p^p([t, T]), i = 1, 2$ be a solution of BSDE$(t, \eta_i, f_i)$.

1. If $E\left[\left(\int_t^T |f_1(s, Y^{2, i}_s, Z^{2, i}_s) - f_2(s, Y^{2, i}_s, Z^{2, i}_s)| \, ds\right)^{\hat{p}}\right] < \infty$ for some $\hat{p} \in (1, p]$, then it holds $P$–a.s. that

$$E\left[\sup_{s \in [t, T]} |Y^{1, i}_s - Y^{2, i}_s|^p\right]_{\mathcal{F}_t} \leq C(T, \hat{p}, \gamma)E\left[|\eta_1 - \eta_2|^\hat{p} + \left(\int_t^T |f_1(s, Y^{2, i}_s, Z^{2, i}_s) - f_2(s, Y^{2, i}_s, Z^{2, i}_s)| \, ds\right)^{\hat{p}}\right]_{\mathcal{F}_t}.$$ (1.5)

2. If $\eta_1 \leq \text{(resp.} \geq \text{)} \eta_2$, $P$–a.s. and if $f_1(s, Y^{2, i}_s, Z^{2, i}_s) \leq \text{(resp.} \geq \text{)} f_2(s, Y^{2, i}_s, Z^{2, i}_s), ds \times dP$–a.s. on $[t, T] \times \Omega$, then it holds $P$–a.s. that $Y^{1, i}_s \leq \text{(resp.} \geq \text{)} Y^{2, i}_s$ for any $s \in [t, T]$. 

2 Stochastic Differential Games with Super-square-integrable Controls

Let \((U, \rho_U)\) and \((V, \rho_V)\) be two separable metric spaces. For some \(u_0 \in U\) and \(v_0 \in V\), we define
\[
[u]_U \overset{\Delta}{=} \rho_U(u, u_0), \quad \forall u \in U \quad \text{and} \quad [v]_V \overset{\Delta}{=} \rho_V(v, v_0), \quad \forall v \in V.
\]

We shall study a zero-sum stochastic differential game between two players, player I and player II, who choose super-square-integrable \(U\)-valued controls and \(V\)-valued controls respectively to compete:

**Definition 2.1.** Given \(t \in [0,T]\), an admissible control process \(\mu = \{\mu_s\}_{s \in [t,T]}\) for player I over period \([t,T]\) is a \(U\)-valued, \(\mathcal{F}\)-progressively measurable process such that \(E \int_t^T [\mu_s]^q \, ds < \infty\) for some \(q > 2\). Admissable control processes for player II over period \([t,T]\) are defined similarly. We denote by \(U_t\) (resp. \(V_t\)) the set of all admissible controls for player I (resp. II) over period \([t,T]\).

**Remark 2.1.** The reason why we use super-square-integrable controls lies in the fact that in the proof of Proposition 2.7, the set of \(U\)-valued (resp. \(V\)-valued) square integrable processes is not closed under Girsanov transformation via functions of the Cameron-Martin space (see in particular [4,16]).

**Lemma 2.1.** Let \(t \in [0,T]\) and \(\tau \in S_{t,T}\). For any \(\mu^1, \mu^2 \in U_t, \mu_s^1 = 1_{\{s < \tau\}}\mu^1_s + 1_{\{s \geq \tau\}}\mu^2_s, \ s \in [t,T]\) defines a \(U_t\)-control. Similarly, for any \(\nu^1, \nu^2 \in V_t, \nu_s^1 = 1_{\{s < \tau\}}\nu^1_s + 1_{\{s \geq \tau\}}\nu^2_s, \ s \in [t,T]\) defines a \(V_t\)-control.

2.1 Game Setting: A Controlled SDE–BSDE System

Our zero-sum stochastic differential game is formulated via a (decoupled) SDE–BSDE system with the following parameters: Fix \(k \in \mathbb{N}, \gamma > 0\) and \(p \in [1,2]\).

1) Let \(b : [0,T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) be a \(\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})\)-measurable function and let \(\sigma : [0,T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{k \times d}\) be a \(\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})\)-measurable function such that for any \((t, u, v) \in [0,T] \times U \times V\) and \((x, x') \in \mathbb{R}^k\)
\[
|b(t, 0, 0, v)| + |\sigma(t, 0, 0, v)| \leq \gamma (1 + [u]_U + [v]_V) \tag{2.1}
\]
and
\[
|b(t, x, u, v) - b(t, x', u, v)| + |\sigma(t, x, u, v) - \sigma(t, x', u, v)| \leq \gamma |x - x'|. \tag{2.2}
\]

2) Let \(g : \mathbb{R}^k \to \mathbb{R}\) be a \(2/p\)-Hölder continuous function with coefficient \(\gamma\).

3) Let \(f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}\) be a \(\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})\)-measurable function such that for any \((t, u, v) \in [0,T] \times U \times V\) and any \((x, y, z), (x', y', z') \in \mathbb{R}^k \times \mathbb{R}^d\)
\[
|f(t, 0, 0, 0, u, v)| \leq \gamma \left( 1 + [u]_U^{2/p} + [v]_V^{2/p} \right) \tag{2.3}
\]
and
\[
|f(t, x, y, z, u, v) - f(t, x', y', z', u, v)| \leq \gamma (|x - x'|^{2/p} + |y - y'| + |z - z'|). \tag{2.4}
\]

For any \(\lambda \geq 0\), we let \(c_\lambda\) denote a generic constant, depending on \(\lambda, T, \gamma, p\) and \(|g(0)|\), whose form may vary from line to line. (In particular, \(c_0\) stands for a generic constant depending on \(T, \gamma, p\) and \(|g(0)|\).)

Also, we would like to introduce two control neutralizers \(\psi, \bar{\psi}\) for the coefficients: For some \(\kappa > 0\)
\((\mathbf{A-u})\) there exist a function \(\psi : [0,T] \times (\mathbb{R} \setminus \mathbb{O}_\kappa(u_0)) \to \mathbb{V}\) that is \(\mathcal{B}([0,T]) \times \mathcal{B}(\mathbb{R} \setminus \mathbb{O}_\kappa(u_0)) \otimes \mathcal{B}(\mathbb{V})\)-measurable and satisfies: for any \((t, x, y, z) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\) and \(u, u' \in \mathbb{O}_\kappa(u_0)\)
\[
b(t, x, u, \psi(t, u)) = b(t, x, u', \psi(t, u')), \quad \sigma(t, x, u, \psi(t, u)) = \sigma(t, x, u', \psi(t, u')), \quad f(t, x, y, z, u, \psi(t, u)) = f(t, x, y, z, u', \psi(t, u')) \and \psi(t, u)]_V \leq \kappa (1 + [u]_U); \tag{2.5}
\]

\((\mathbf{A-v})\) and there exists a function \(\bar{\psi} : [0,T] \times (\mathbb{V} \setminus \mathbb{O}_\kappa(v_0)) \to \mathbb{U}\) that is \(\mathcal{B}([0,T]) \times \mathcal{B}(\mathbb{V} \setminus \mathbb{O}_\kappa(v_0)) \otimes \mathcal{B}(\mathbb{V})\)-measurable and satisfies: for any \((t, x, y, z) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\) and \(v, v' \in \mathbb{V} \setminus \mathbb{O}_\kappa(v_0)\)
\[
b(t, x, \bar{\psi}(t, v), v) = b(t, x, \bar{\psi}(t, v'), v'), \quad \sigma(t, x, \bar{\psi}(t, v), v) = \sigma(t, x, \bar{\psi}(t, v'), v'), \quad f(t, x, y, z, \bar{\psi}(t, v), v) = f(t, x, y, z, \bar{\psi}(t, v'), v') \and \bar{\psi}(t, v)]_U \leq \kappa (1 + [v]_V).
\]
Remark 2.2. A typical example satisfying both (A-u) and (A-v) is the additive-control case: Let \( U = V = \mathbb{R}^l \) and consider the following coefficients:

\[
\begin{align*}
    b(t, x, u, v) &= b(t, x, u + v), \quad \sigma(t, x, u, v) = \sigma(t, x, u + v) \\
    f(t, x, y, z, u, v) &= f(t, x, y, z, u + v), \quad \forall (t, x, y, z, u, v) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^d \times U \times V.
\end{align*}
\]

Then (A-u) and (A-v) hold for functions \( \psi(u) = -u \) and \( \bar{\psi}(v) = -v \) respectively.

When the game begins at time \( t \in [0, T] \), player I and player II select admissible controls \( \mu \in \mathcal{U}_t \) and \( \nu \in \mathcal{V}_t \) respectively. Then the state process starting from \( \xi \in L^2(\mathcal{F}_t, \mathbb{R}^k) \) will evolve according to SDE (1.1) on the probability space \((\Omega, \mathcal{F}_t, P)\). The measurability of functions \( b, \sigma, \mu \) and \( \nu \) implies that

\[
b^{\mu, \nu}(s, \omega, x) \overset{\Delta}{=} b(s, x, \mu_s(\omega), \nu_s(\omega)), \quad \forall (s, \omega, x) \in [t, T] \times \Omega \times \mathbb{R}^k
\]

is \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^k) / \mathcal{B}(\mathbb{R}^k) \)-measurable and that

\[
\sigma^{\mu, \nu}(s, \omega, x) \overset{\Delta}{=} \sigma(s, x, \mu_s(\omega), \nu_s(\omega)), \quad \forall (s, \omega, x) \in [t, T] \times \Omega \times \mathbb{R}^k
\]

is \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^k) / \mathcal{B}(\mathbb{R}^{k \times d}) \)-measurable. Also, (2.2) (2.1) and Hölder’s inequality show that \( b^{\mu, \nu}, \sigma^{\mu, \nu} \) are Lipschitz continuous in \( x \) and satisfy

\[
E \left[ \left( \int_t^T |b^{\mu, \nu}(s, 0)| ds \right)^2 \right] \leq c_0 + c_0 E \left[ \left( \int_t^T |\sigma^{\mu, \nu}(s, 0)| ds \right)^2 \right] < \infty.
\]

Then it is well-known that the SDE (1.1) admits a unique solution \( \{X_t^{\mu, \nu} \}_{s \in [t, T]} \in C^2([t, T], \mathbb{R}^k) \) such that

\[
E \left[ \sup_{s \in [t, T]} |X_s^{\mu, \nu}|^2 \right] \leq c_0 E[|\xi|^2] + c_0 E \left[ \left( \int_t^T |b^{\mu, \nu}(s, 0)| ds \right)^2 \right] + E \left[ \left( \int_t^T |\sigma^{\mu, \nu}(s, 0)| ds \right)^2 \right] < \infty.
\]

(2.5)

Given \( s \in [t, T] \), let \( [\mu]_s \) denote the restriction of \( \mu \) over period \( [s, T] \): i.e., \( [\mu]_r \overset{\Delta}{=} \mu_r, \forall r \in [s, T] \). Clearly, \( [\mu]_s \in \mathcal{U}_s \), similarly, \( \{[\nu]_r \overset{\Delta}{=} \nu_r \}_{r \in [s, T]} \in \mathcal{V}_s \).

As

\[
X_r^{\xi, \mu, \nu} = X_s^{\xi, \mu, \nu} + \int_s^r b\left(r', X_r^{\xi, \mu, \nu}, \mu_r, \nu_r \right) dr' + \int_s^r \sigma\left(r', X_r^{\xi, \mu, \nu}, \mu_r, \nu_r \right) dB_r
\]

we see that \( \{X_t^{\xi, \mu, \nu} \}_{r \in [s, T]} \in C^2(\mathbb{R}^k) \) solves (1.1) with the parameters \( \mu^s, [\nu]_s \). To wit, it holds \( P \)-a.s. that

\[
X_r^{\xi, \mu, \nu} = X_r^{\xi, \mu, \nu}^{[\mu]^s,[\nu]^s}, \quad \forall r \in [s, T].
\]

(2.6)

Lemma 2.2. Given \( t \in [0, T] \), let \( \xi \in L^2(\mathcal{F}_t, \mathbb{R}^k) \) and \((\mu, \nu) \in \mathcal{U}_t \times \mathcal{V}_t \). If \((\mu, \nu) = (\bar{\mu}, \bar{\nu})\), \( dr \times dP \)-a.s. on \([t, \tau[ \cup [\tau, T])_A \) for some \( \tau \in S_{t,T} \) and \( A \in \mathcal{F}_T \), then it holds \( P \)-a.s. that

\[
1_A X_s^{\xi, \mu, \nu} + 1_{A'} X_s^{\xi, \bar{\mu}, \bar{\nu}} = 1_A X_s^{\xi, \bar{\mu}, \bar{\nu}} + 1_{A'} X_s^{\xi, \bar{\mu}, \bar{\nu}}, \quad \forall s \in [t, T].
\]

(2.7)

Now, let \( \Theta \) stand for the quadruplet \((t, \xi, \mu, \nu)\). Given \( \tau \in S_{t,T} \), the measurability of \((f, X^{\Theta}, \mu, \nu)\) and (2.4) imply that

\[
f_\tau^{\Theta}(s, \omega, y, z) \overset{\Delta}{=} 1_{\{s < \tau(\omega)\}} f \left(s, X_s^{\Theta}(\omega), y, z, \mu_s(\omega), \nu_s(\omega) \right), \quad \forall (s, \omega, y, z) \in [t, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d
\]
Lemma 2.3. and have the following a priori estimate:

\[
E \left[ \left( \int_t^T |f^\Theta_r(s, 0, 0)| ds \right)^p \right] \leq c_0 + c_0 E \left[ \sup_{s \in [t, T]} |X^\Theta_s|^2 + \int_t^T (|\mu_s|^2 + |\nu_s|^2) ds \right] < \infty.
\] (2.8)

Thus, for any \( \eta \in L^p(F_t) \), Proposition 1.1 shows that the BSDE \( (t, \eta, f^\Theta_t) \) admits a unique solution \( (Y^\Theta(\tau, \eta), Z^\Theta(\tau, \eta)) \in G^p_F([t, T]) \), which has the following estimate as a consequence of (1.5).

Corollary 2.1. Let \( t \in [0, T) \), \( \xi \in L^2(F_t, \mathbb{R}^k) \), \( (\mu, \nu) \in \mathcal{U}_t \times \mathcal{V}_t \) and \( \tau \in \mathcal{S}_{t,T} \). Given \( \eta_1, \eta_2 \in L^p(F_t) \), it holds for any \( \bar{p} \in (1, p] \) that

\[
E \left[ \sup_{s \in [t, T]} \left| Y^t,\xi,\mu,\nu_s(\tau, \eta_1) - Y^t,\xi,\mu,\nu_s(\tau, \eta_2) \right|^\bar{p} \right] \leq c_\bar{p} E \left[ |\eta_1 - \eta_2|^\bar{p} \right], \quad P\text{-a.s.}
\] (2.9)

Given another stopping time \( \zeta \in \mathcal{S}_{t,T} \) with \( \zeta \leq \tau \), P-a.s., one can easily show that \( \left\{ (Y^\Theta_{\zeta \wedge s}(\tau, \eta), 1_{\{s < \zeta\}} Z^\Theta_{\zeta \wedge s}(\tau, \eta)) \right\}_{s \in [t, T]} \in \mathbb{G}^p_F([t, T]) \) solves the BSDE \( (t, Y^\Theta(\tau, \eta), f^\Theta_\zeta) \). To wit, we have

\[
\left( Y^\Theta_\zeta(\zeta, Y_\zeta^\Theta(\tau, \eta)), Z_\zeta^\Theta(\zeta, Y_\zeta^\Theta(\tau, \eta)) \right) = \left( Y^\Theta_{\zeta \wedge s}(\tau, \eta), 1_{\{s < \zeta\}} Z^\Theta_{\zeta \wedge s}(\tau, \eta) \right), \quad s \in [t, T].
\] (2.10)

In particular, when \( \zeta = \tau \),

\[
\left( Y^\Theta(\tau, \eta), Z^\Theta(\tau, \eta) \right) = \left( Y^\Theta_{\tau \wedge s}(\tau, \eta), 1_{\{s < \tau\}} Z^\Theta_{\tau \wedge s}(\tau, \eta) \right), \quad s \in [t, T].
\] (2.11)

On the other hand, if \( \tau \in \mathcal{S}_{S_{t,T}} \) for some \( s \in [t, T] \), letting \( \Theta^s \overset{\triangle}{=} (s, X^\Theta_s, [\mu]^s, [\nu]^s) \), we can deduce from (2.6) that \( \left\{ (Y^\Theta_s(\tau, \eta), Z^\Theta_s(\tau, \eta)) \right\}_{r \in [s, T]} \in \mathbb{G}^p_F([s, T]) \) solves the following BSDE \( (s, \eta, f^\Theta_s) \):

\[
Y_s = \eta + \int_r^T 1_{\{r' < \tau\}} f(r', X^\Theta_{r'}, Y_{r'}, Z_{r'}, [\mu]_{r'}, [\nu]_{r'}) dr' - \int_r^T Z_{r'} dB_r
\]

\[
= \eta + \int_r^T 1_{\{r' < \tau\}} f(r', X^\Theta_{r'}, Y_{r'}, Z_{r'}, [\mu]_{r'}, [\nu]_{r'}) dr' - \int_r^T Z_{r'} dB_r, \quad r \in [s, T].
\]

Hence, it holds P-a.s. that

\[
Y^\Theta_r(\tau, \eta) = Y^{\Theta r}_r(\tau, \eta), \quad \forall r \in [s, T].
\] (2.12)

The \( 2/p \)-Hölder continuity of functions \( g \) and (2.5) show that \( g(X^\Theta_T) \in L^p(F_T) \). We set \( J(\Theta) \overset{\triangle}{=} Y^{\Theta T}_t(\tau, g(X^\Theta_T)) \) and have the following a priori estimate:

Lemma 2.3. Let \( t \in [0, T) \) and \( (\mu, \nu) \in \mathcal{U}_t \times \mathcal{V}_t \). Given \( \xi_1, \xi_2 \in L^2(F_t, \mathbb{R}^k) \), it holds for any \( \bar{p} \in (1, p] \) that

\[
E \left[ \sup_{s \in [t, T]} \left| Y^t,\xi_1,\mu,\nu_s(T, g(X^\Theta_T)) - Y^t,\xi_2,\mu,\nu_s(T, g(X^\Theta_T)) \right|^\bar{p} \right] \leq c_\bar{p} |\xi_1 - \xi_2|^{\bar{p}/\bar{p}}, \quad P\text{-a.s.}
\] (2.13)

2.2 Definition of Value Functions and a Weak Dynamic Programming Principle

Now, we are ready to introduce values of the zero-sum stochastic differential games via the following version of Elliott–Kalton strategies (or non-anticipative strategies).

Definition 2.2. Given \( t \in [0, T] \), an admissible strategy \( \alpha \) for player I over period \( [t, T] \) is a mapping \( \alpha : \mathcal{V}_t \rightarrow \mathcal{U}_t \) satisfying: (i) There exists a \( C_\alpha > 0 \) such that for any \( \nu \in \mathcal{V}_t \), \( (\alpha(\nu))_{s,t} \leq C_\alpha [\nu]_{s,t}, ds \times dP \text{-a.s.} \), where \( \kappa \) is the constant appeared in (A-u) and (A-v); (ii) For any \( \nu_1^{s}, \nu_2^{s} \in \mathcal{V}_t, \tau \in \mathcal{S}_{t,T} \) and \( \mathcal{A} \in F_T, \) if \( \nu_1 = \nu_2, ds \times dP \text{-a.s.} \) on \( [t, T] \cup [\tau, T] \), then \( \alpha(\nu_1) = \alpha(\nu_2), ds \times dP \text{-a.s.} \) on \( [t, T] \cup [\tau, T] \).

Admissible strategies \( \beta : \mathcal{U}_t \rightarrow \mathcal{V}_t \) for player II over period \( [t, T] \) are defined similarly. The collection of all admissible strategies for player I (resp. II) over period \( [t, T] \) is denoted by \( \mathfrak{A}_t \) (resp. \( \mathfrak{B}_t \)).
Remark 2.3. The condition (ii) of Definition 2.2 is called the nonanticipativity of strategies. It is said in [10, line 4 of page 456] that “From the nonanticipativity of \( \beta_2 \) we have \( \beta_2(u^*_2) = \sum_{j \geq 1} 1_{\Delta_j} \beta_2(u^*_j), \cdots \)”. What actually used in this equality is not the nonanticipativity of \( \beta_2 \) as defined in Definition 3.2 therein, but the requirement:

For any \( u, u' \in U_{t+\delta} \) and \( A \in F_{t+\delta} \), if \( u = u' \) on \( [t+\delta, T] \times A \), then \( \beta_2(u) = \beta_2(u') \) on \( [t+\delta, T] \times A \).

Since \( \beta_2 \) is a restriction of strategy \( \beta \in B_{t,T} \) over period \( [t+\delta, T] \), (2.14) entails the following condition on \( \beta \).

For any \( u, u' \in U_{t,T} \), any \( s \in [t, T] \) and any \( A \in F_s \), if \( u = u' \) on \( ([t, s] \times \Omega) \cup ([s, T] \times A) \), then \( \beta(u) = \beta(u') \) on \( ([t, s] \times \Omega) \cup ([s, T] \times A) \).

which is exactly a simple version of our nonanticipativity condition on strategies with \( \tau = s \).

For any \( (t, x) \in [0, T] \times \mathbb{R}^k \), we define

\[
\begin{align*}
\text{w}_1(t, x) &\triangleq \underset{\beta \in \mathcal{B}_t}{\text{essinf}} \ \underline{\text{sup}} \ J(t, x, \mu, \beta(\mu)) = \underset{\beta \in \mathcal{B}_t}{\text{essinf}} \ \underline{\text{sup}} \ Y^{t,x,\mu,\beta(\mu)}_t(T, g(X^{t,x,\mu,\beta(\mu)}_t)) \\
\text{w}_2(t, x) &\triangleq \underset{\alpha \in \mathcal{A}_t, \ \nu \in \mathcal{V}_t}{\text{esssup}} \ J(t, x, \alpha(\nu), \nu) = \underset{\alpha \in \mathcal{A}_t, \ \nu \in \mathcal{V}_t}{\text{esssup}} \underline{\text{essinf}} \ Y^{t,x,\alpha(\nu),\nu}_t(T, g(X^{t,x,\alpha(\nu),\nu}_t))
\end{align*}
\]

as player I’s and player II’s priority values of the zero-sum stochastic differential game that starts from time \( t \) with initial state \( x \).

Remark 2.4. When \( f \) is independent of \((y, z)\), \( w_1 \) and \( w_2 \) are in form of

\[
\begin{align*}
\text{w}_1(t, x) &\triangleq \underset{\beta \in \mathcal{B}_t}{\text{essinf}} \ \underline{\text{sup}} \ E \left[ g(X^{t,x,\mu,\beta(\mu)}_T) + \int_t^T f(s, X^{t,x,\mu,\beta(\mu)}_s, \mu_s, (\beta(\mu))_s) ds \right]_{\mathcal{F}_t} \\
\text{w}_2(t, x) &\triangleq \underset{\alpha \in \mathcal{A}_t, \ \nu \in \mathcal{V}_t}{\text{esssup}} \underline{\text{essinf}} \ E \left[ g(X^{t,x,\alpha(\nu),\nu}_T) + \int_t^T f(s, X^{t,x,\alpha(\nu),\nu}_s, (\alpha(\nu))_s, \nu_s) ds \right]_{\mathcal{F}_t}, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^k.
\end{align*}
\]

Remark 2.5. When \( U \) and \( V \) are compact (say \( U = \overline{\mathcal{U}}_\kappa(w_0) \) and \( V = \overline{\mathcal{V}}_\kappa(v_0) \)), Assumptions (A-u), (A-v) are no longer needed, and the integrability condition in Definition 2.1 as well as the condition (i) in Definition 2.2 hold automatically. Thus our game problem degenerates to the case of [10].

Let us review some basic properties of the essential extrema for the later use (see e.g. [23, Proposition VI-1-1] or [10, Theorem A.3.2]):

Lemma 2.4. Let \( \{\xi_i\}_{i \in I}, \ \{\eta_i\}_{i \in I} \) be two classes of \( \mathcal{F}_T \)–measurable random variables with the same index set \( I \).

1. If \( \xi_i \leq (\geq) \eta_i \), \( P \)-a.s. holds for all \( i \in I \), then \( \text{esssup} \xi_i \leq (\geq) \text{esssup} \eta_i \), \( P \)-a.s. 

2. For any \( A \in \mathcal{F}_T \), it holds \( P \)-a.s. that \( \text{esssup}_{i \in I} (1_A \xi_i + 1_A^c \eta_i) = 1_A \text{esssup}_{i \in I} \xi_i + 1_A^c \text{esssup}_{i \in I} \eta_i \). In particular, \( \text{esssup}_{i \in I} (1_A \xi_i) = 1_A \text{esssup}_{i \in I} \xi_i, \ P \)-a.s. 

3. For any \( \mathcal{F}_T \)-measurable random variable \( \eta \) and any \( \lambda > 0 \), we have \( \text{esssup}_{i \in I} (\lambda \xi_i + \eta) = \lambda \text{esssup}_{i \in I} \xi_i + \eta, \ P \)-a.s.

(1)-(3) also hold when we replace \( \text{esssup} \) by \( \text{essinf} \).

The values \( w_1, w_2 \) are bounded as follows:

Proposition 2.1. For any \( (t, x) \in [0, T] \times \mathbb{R}^k \), it holds \( P \)-a.s. that \( |w_1(t, x)| + |w_2(t, x)| \leq c_\kappa + c_0 |x|^{2/p} \).

Similar to Proposition 3.1 of [10], the following result allows us to regard \( w_1 \) and \( w_2 \) as deterministic functions on \([0, T] \times \mathbb{R}^k\):

Proposition 2.2. Let \( i = 1, 2 \). For any \( (t, x) \in [0, T] \times \mathbb{R}^k \), it holds \( P \)-a.s. that \( w_i(t, x) = E[w_i(t, x)] \).

Moreover, \( w_1 \) and \( w_2 \) are \( 2/p \)-Hölder continuous in \( x \):
Proposition 2.3. For any \( t \in [0,T] \) and \( x_1, x_2 \in \mathbb{R}^k \), \( |w_1(t,x_1) - w_1(t,x_2)| + |w_2(t,x_1) - w_2(t,x_2)| \leq c_0|x_1 - x_2|^{2/p} \).

However, the values \( w_1, w_2 \) are generally not continuous in \( t \) unless \( U, V \) are compact.

Remark 2.6. When trying to directly prove the dynamic programming principle, [13] encountered a measurability issue: The pasted strategies for approximation may not be progressively measurable, see page 299 therein. So they first proved that the value functions are unique viscosity solutions to the associated Bellman-Isaacs equations by a time-discretization approach (assuming that the limiting Isaacs equation has a comparison principle), which relies on the following regularity of the approximating values \( v_\pi \):

\[
|v_\pi(t,x) - v_\pi(t',x')| \leq C(|t - t'|^{1/2} + |x - x'|) \quad \forall (t,x), (t',x') \in [0,T] \times \mathbb{R}^k
\]

with a uniform coefficient \( C > 0 \) for all partitions \( \pi \) of \([0,T]\). Since our value functions \( w_1, w_2 \) may not be \( \frac{1}{2} - \) Hölder continuous in \( t \), this method seems not suitable for our problem. Hence, we adopt Buckdahn and Li’s probability setting.

The following weak dynamic programming principle for value functions \( w_1, w_2 \) is the main topic of the paper:

**Theorem 2.1.** 1) Given \( t \in [0,T] \), let \( \phi, \overline{\phi} : [t,T] \times \mathbb{R}^k \to \mathbb{R} \) be two continuous functions such that \( \phi(s,x) \leq w_1(s,x) \leq \overline{\phi}(s,x), (s,x) \in [t,T] \times \mathbb{R}^k \). Then for any \( x \in \mathbb{R}^k \) and \( \delta \in (0,T-t) \), it holds \( P-\text{a.s.} \)

\[
\text{essinf}_{\beta \in \mathbb{A}_t} \text{esssup}_{\mu \in \mathcal{U}_t} Y^t_{x,\mu}(\tau_{\beta,\mu}, X^t_{x,\mu}) \leq w_1(t,x) \leq \text{essinf}_{\beta \in \mathbb{A}_t} \text{esssup}_{\mu \in \mathcal{U}_t} Y^t_{x,\mu}(\tau_{\beta,\mu}, \overline{\phi}(X^t_{x,\mu}))
\]

where \( \tau_{\beta,\mu} \triangleq \inf \{ s \in (t,T) : (s,X^s_{x,\mu}) \notin O(\delta) \} \).

2) Given \( t \in [0,T] \), let \( \phi, \overline{\phi} : [t,T] \times \mathbb{R}^k \to \mathbb{R} \) be two continuous functions such that \( \phi(s,x) \leq w_2(s,x) \leq \overline{\phi}(s,x), (s,x) \in [t,T] \times \mathbb{R}^k \). Then for any \( x \in \mathbb{R}^k \) and \( \delta \in (0,T-t) \), it holds \( P-\text{a.s.} \)

\[
\text{esssup}_{\alpha \in \mathbb{A}_t} \text{essinf}_{\nu \in \mathcal{V}_t} Y^t_{x,\nu}(\tau_{\alpha,\nu}, \phi(X^t_{x,\nu})) \leq w_2(t,x) \leq \text{esssup}_{\alpha \in \mathbb{A}_t} \text{essinf}_{\nu \in \mathcal{V}_t} Y^t_{x,\nu}(\tau_{\alpha,\nu}, \overline{\phi}(X^t_{x,\nu}))
\]

where \( \tau_{\alpha,\nu} \triangleq \inf \{ s \in (t,T) : (s,X^s_{x,\nu}) \notin O(\delta) \} \).

3 Viscosity Solutions of Related Fully Non-linear PDEs

In this section, we show that the priority values are (discontinuous) viscosity solutions to the following partial differential equation with a fully non-linear Hamiltonian \( H \):

\[
- \frac{\partial}{\partial t} w(t,x) - H(t,x,w(t,x), Dw(t,x), D^2_w(t,x)) = 0, \quad \forall (t,x) \in (0,T) \times \mathbb{R}^k.
\]

**Definition 3.1.** Let \( H : [0,T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{S}_k \to [-\infty, \infty] \) be an upper (resp. lower) semicontinuous functions with \( \mathbb{S}_k \) denoting the set of all \( k \times k \)-valued symmetric matrices. An upper (resp. lower) semicontinuous function \( w : [0,T] \times \mathbb{R}^k \to \mathbb{R} \) is called a viscosity subsolution (resp. supersolution) of (3.1) if for any \( (t_0,x_0, \varphi) \in (0,T) \times \mathbb{R}^k \times C^{1,2}(0,T] \times \mathbb{R}^k \) such that \( w(t_0,x_0) = \varphi(t_0,x_0) \) and that \( w - \varphi \) attains a strict local maximum (resp. strict local minimum) at \( (t_0,x_0) \), we have

\[
- \frac{\partial}{\partial t} \varphi(t_0,x_0) - H(t_0,x_0, \varphi(t_0,x_0), D\varphi(t_0,x_0), D^2\varphi(t_0,x_0)) \leq (\text{resp.} \geq) 0.
\]

For any \( (t,x,y,z,\Gamma,u,v) \in [0,T] \times \mathbb{R}^k \times \mathbb{R}^d \times \mathbb{S}_k \times \mathbb{U} \times \mathbb{V} \), we set

\[
H(t,x,y,z,\Gamma,u,v) \overset{\triangle}{=} \frac{1}{2} \text{trace} (\sigma \sigma^T(t,x,u,v) \Gamma) + z \cdot b(t,x,u,v) + f(t,x,y,z, \sigma(t,x,u,v), u,v)
\]
and consider the following Hamiltonian functions:

\[
H_1(\Xi) \triangleq \sup_{u \in U} \lim_{n \to \infty} \inf_{v \in V} H(\Xi', u, v), \quad \overline{H}_1(\Xi) \triangleq \lim_{n \to \infty} \sup_{u \in U} \lim_{v \to u} \inf_{v \in V} H(\Xi, u', v),
\]

and \( \overline{H}_2(\Xi) \triangleq \inf_{v \in V} \lim_{n \to \infty} \sup_{u \in U} H(\Xi', u, v), \) where \( \Xi = (t, x, y, z) \), \( \mathcal{O}_u^n \triangleq \{ v \in V : [v]_y \leq \kappa + n[u]_y \} \), \( \mathcal{O}_v^n \triangleq \{ u \in U : [u]_u \leq \kappa + n[v]_y \} \), \( \mathcal{O}_u \triangleq \bigcup_{n \in \mathbb{N}} \mathcal{O}_u^n \).

**Remark 3.1.** When \( U \) and \( V \) are compact (say \( U = \overline{G}(u_0) \) and \( V = \overline{G}(v_0) \)), it holds for any \((u, v) \in U \times V \) and \( n \in \mathbb{N} \) that \((\mathcal{O}_u^n, \mathcal{O}_v^n) = (V, U)\). If further assuming as \( \overline{H}_1 \) that for any \((x, y, z) \in \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^d \), \( \sigma(\cdot, x, y, z, \cdot), f(\cdot, x, y, z, \cdot) \) are all continuous in \((t, u, v)\), one can deduce from \( (2.1) - (2.4) \) that the continuity of \( H(\Xi, u, v) \) in \( \Xi \) is uniform in \((u, v)\). It follows that

\[
H_1(\Xi) = \sup_{u \in U} \lim_{v \to u} \inf_{v \in V} H(\Xi', u, v) = \sup_{u \in U} \lim_{v \to u} H(\Xi, u, v),
\]

and that

\[
\overline{H}_1(\Xi) = \lim_{n \to \infty} \sup_{u \in U} \lim_{v \to u} \inf_{v \in V} H(\Xi', u, v) = \sup_{u \in U} \inf_{v \in V} H(\Xi, u, v) = \overline{H}(\Xi).
\]

Similarly, \( \overline{H}_2(\Xi) = \overline{H}(\Xi) \).

**Theorem 3.1.** For \( i = 1, 2 \), \( \underline{w}_i \) (resp. \( \overline{w}_i \)) is a viscosity supersolution (resp. subsolution) of \( (3.1) \) with the fully nonlinear Hamiltonian \( \underline{H}_i \) (resp. \( \overline{H}_i \)).

**Remark 3.2.** Given \( i = 1, 2 \) and \( x \in \mathbb{R}^k \), in spite of \( w_i(T, x) = g(x) \), it is possible that neither \( \underline{w}_i(T, x) \) nor \( \overline{w}_i(T, x) \) equals to \( g(x) \) since \( w_i \) may not be continuous in \( t \).

## 4 Proofs

**Proof of Proposition 1.1:** Set \( \mathcal{F}(s, \omega, y, z) \triangleq 1_{\{s \geq t\}} f(s, \omega, y, z) \), \( \forall (s, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \). Clearly, \( \mathcal{F} \) is also a \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) / \mathcal{B}(\mathbb{R}) \)-measurable function Lipschitz continuous in \((y, z)\). As \( E\left[ (\int_t^T |f(s, 0, 0)| ds)^p \right] = E\left[ (\int_t^T f(s, 0, 0) ds)^p \right] < \infty \), Theorem 4.2 of [1] shows that the BSDE

\[
Y_s = \eta + \int_s^T f(r, Y_r, Z_r) \, dr - \int_s^T Z_r dB_r, \quad s \in [0, T].
\]

admits a unique solution \((Y, Z) \in \mathbb{G}_F^p([0, T])\). In particular, \( \{(Y_s, Z_s)\}_{s \in [t, T]} \in \mathbb{G}_F^p([t, T]) \) solves \( (1.3) \).

Suppose that \((Y', Z')\) is another solution of \( (1.3) \) in \( \mathbb{G}_F^p([t, T]) \). Let \((\tilde{Y}', \tilde{Z}') \in \mathbb{G}_F^p([0, t])\) be the unique solution of the following BSDE with zero generator:

\[
\tilde{Y}'_s = Y'_t - \int_s^t \tilde{Z}'_r dB_r, \quad s \in [0, t].
\]
Actually, \( \bar{Y}' = E[Y'|\mathcal{F}_s] \). Then \((Y', Z') = (Y_s + (Y', Z') = (Y_s, Z_s, \forall s \in [t, T].

Given \( A \in \mathcal{F}_s \), multiplying \( 1_A \) to both sides of (1.3) yields that
\[
1_A Y_s = 1_A \eta + \int_s^T 1_A f(r, 1_A Y_r, 1_A Z_r) dr - \int_s^T 1_A Z_r dB_r, \quad s \in [t, T].
\]
Let \( (Y^A, Z^A) \in \mathbb{G}_F([0, t]) \) be the unique solution of the following BSDE with zero generator:
\[
Y^A_s = 1_A \eta - \int_s^T Z^A_r dB_r, \quad s \in [0, t].
\]
Then \((Y^A, Z^A) \triangleq \{ (1_{\{s<t\}} Y^A_s + 1_{\{s\geq t\}} Y_s, 1_{\{s<t\}} Z^A_s + 1_{\{s\geq t\}} Z_s) \}_{s \in [0, T]} \in \mathbb{G}_F([0, T]) \) solves the BSDE
\[
Y^A_s = 1_A \eta + \int_s^T f_A(r, Y^A_r, Z^A_r) dr - \int_s^T Z^A_r dB_r, \quad s \in [0, T],
\]
where \( f_A(r, \omega, y, z) \triangleq 1_{\{r \geq t\}} f_{\omega}(r, y, z) \). Since \( \{1_{\{r \geq t\}} \}_{r \in [0, T]} \) is a right-continuous \( \mathbb{F} \)-adapted process, the measurability and Lipschitz continuity of \( f \) imply that \( f_A \) is also a \( \mathcal{F} \otimes \mathcal{B}d \) measurable function Lipschitz continuous in \((y, z)\). Since \( E \left[ \left( \int_0^T |f_A(s, 0, 0)| ds \right)^p \right] \leq E \left[ \left( \int_0^T |f_A(s, 0, 0)| ds \right)^p \right] \leq E \left[ \left( \int_0^T |f_A(s, 0, 0)| ds \right)^p \right] < \infty \), applying Proposition 3.2 of \( [7] \) yields that
\[
E \left[ 1_A \sup_{s \in [t, T]} |Y^A_s|^p \right] \leq E \left[ 1_A \sup_{s \in [0, T]} |Y^A_s|^p \right] \leq C(T, p, \gamma) E \left[ |1_A \eta|^p + \left( \int_0^T |f_A(s, 0, 0)| ds \right)^p \right],
\]
where \( f_A(r, \omega, y, z) \triangleq f_1(r, \omega, y, z) + f_2(s, Y^2, Z^2) \). Clearly, \( \tilde{f}_s \) is a \( \mathcal{F} \otimes \mathcal{B}(d) \) measurable function Lipschitz continuous in \((y, z)\). Suppose that \( E \left[ \left( \int_0^T |\tilde{f}(s, 0, 0)| ds \right)^p \right] < \infty \) for some \( p \in (1, p) \). Since \( \mathbb{G}_F([0, t]) \subset \mathbb{G}_F([t, T]) \) by Hölder’s inequality, applying Proposition 1.1 with \( p = \tilde{p} \) shows that \((\tilde{Y}, \tilde{Z})\) is the unique solution of BSDE\((t, \eta_1 - \eta_2, \tilde{f})\) in \( \mathbb{G}_F([t, T]) \) satisfying
\[
E \left[ \sup_{s \in [t, T]} |\tilde{Y}_s|^p \right] \leq C(T, \tilde{p}, \gamma) E \left[ |\eta_1 - \eta_2|^p + \left( \int_0^T |\tilde{f}(s, 0, 0)| ds \right)^p \right], \quad P-\text{a.s.},
\]
which is exactly (1.5).

(2) Next, suppose that \( \eta_1 \leq (\text{resp.} \geq) \eta_2, P-\text{a.s.} \) and that \( \delta f_s \triangleq f_1(s, Y^1_s, Z^1_s) - f_2(s, Y^2_s, Z^2_s) \leq (\text{resp.} \geq) 0 \), \( ds \times dP-\text{a.s.} \) on \([t, T] \times \Omega \). By (2.4),
\[
ad_s \triangleq 1_{\{\tilde{Y}_s \neq 0\}} \frac{f_1(s, Y^1_s, Z^1_s) - f_1(s, Y^2_s, Z^2_s)}{Y_s} \in [-\gamma, \gamma], \quad s \in [t, T]
\]
defines an \( \mathbb{F} \)-progressively measurable bounded process. For \( i = 1, \ldots, d \), analogous to process \( a \)
\[
b_s \triangleq 1_{\{Z^1_s \neq Z^2_s\}} \frac{1}{Z^1_s - Z^2_s} \left( f_1(s, Y^2_s, (Z^2_s), Z^2_{s+i-1}, Z^1_{s+i}, \ldots, Z^1_n) - f_1(s, Y^2_s, (Z^2_s), Z^2_{s+i+1}, \ldots, Z^1_n) \right) \in [-\gamma, \gamma], \quad s \in [t, T]
\]
also defines an $\hat{F}$–progressively measurable bounded process.

Then we can alternatively express (4.2) as

$$\hat{Y}_s = \eta_1 - \eta_2 + \int_s^T (a_r \hat{Y}_r + b_r \hat{Z}_r + \delta f_r) \, dr - \int_s^T \hat{Z}_r \, dB_r, \quad s \in [t,T].$$

Define $Q_s \overset{\triangle}{=} \exp \left\{ \int_t^s a_r \, dr - \frac{1}{2} \int_t^s |b_r|^2 \, dr + \int_t^s b_r \, dB_r \right\}$, $s \in [t,T]$. Applying integration by parts yields that

$$Q_s \hat{Y}_s = Q_T \hat{Y}_T + \int_s^T Q_r (a_r \hat{Y}_r + b_r \hat{Z}_r + \delta f_r) \, dr - \int_s^T Q_r \hat{Z}_r \, dB_r - \int_s^T \hat{Y}_r Q_r a_r \, dr - \int_s^T \hat{Y}_r Q_r b_r \, dB_r - \int_s^T Q_r b_r \, \hat{Z}_r \, dr$$

$$= Q_T (\eta_1 - \eta_2) + \int_s^T Q_r \delta f_r \, dr - \int_s^T Q_r (\hat{Y}_r + \hat{Y}_r b_r) \, dB_r, \quad P\text{-a.s.} \tag{4.3}$$

One can deduce from the Burkholder-Davis-Gundy inequality and Hölder’s inequality that

$$E \left[ \sup_{s \in [t,T]} \left| \int_t^s Q_r (\hat{Y}_r + \hat{Y}_r b_r) \, dB_r \right| \right] \leq c_0 E \left[ \left( \int_t^T \left| \int_t^r Q_s \hat{Y}_r \, dB_r \right|^2 \, dr \right)^{\frac{1}{2}} \right] \leq c_0 E \left[ \sup_{s \in [t,T]} |Q_r| \right] \left( \sup_{s \in [t,T]} |\hat{Y}_r| + \left( \int_t^T |\hat{Z}_r|^2 \, dr \right)^{\frac{1}{2}} \right) \right]$$

$$\leq c_0 \left( E \left[ \sup_{s \in [t,T]} |Q_r|^p \right] \right)^{1/p} \left( \|\hat{Y}_r\|_{C^r_p([t,T])} + \|\hat{Z}_r\|_{L^p([t,T], N^d)} \right), \tag{4.4}$$

where $\hat{p} = \frac{p}{p+1}$. Also, Doob’s martingale inequality implies that

$$E \left[ \sup_{s \in [t,T]} |Q_r|^p \right] \leq c_0 E \left[ |Q_r|^p \right] = c_0 E \left[ \exp \left\{ \hat{p} \int_t^T a_r \, dr + \frac{\hat{p}^2 - 1}{2} \int_t^T |b_r|^2 \, dr - \frac{\hat{p}^2}{2} \int_t^T |b_r|^2 \, dr + \hat{p} \int_t^T b_r \, dB_r \right\} \right]$$

$$\leq c_0 \exp \left\{ \frac{\hat{p}}{\gamma} \int_t^T b_r \, dB_r \right\} \exp \left\{ - \frac{\hat{p}^2}{2} \int_t^T |b_r|^2 \, dr + \frac{\hat{p}^2}{2} \int_t^T b_r \, dB_r \right\} \leq c_0 \exp \left\{ \frac{\gamma T + \hat{p}^2 - 1}{2} \gamma^2 T \right\},$$

which together with (4.4) shows that $\left\{ \int_t^s Q_r (\hat{Y}_r b_r + \hat{Z}_r) \, dB_r \right\}_{s \in [t,T]}$ is a uniformly integrable martingale. Then for any $s \in [t,T]$, taking $E[\cdot | \mathcal{F}_s]$ in (4.3) yields that $P$–a.s.

$$Q_s \hat{Y}_s = E \left[ Q_T (\eta_1 - \eta_2) + \int_s^T Q_r \delta f_r \, dB_r \mid \mathcal{F}_s \right] \leq (\text{resp.} \geq) 0, \quad \text{thus } \hat{Y}_s \leq (\text{resp.} \geq) 0.$$

By the continuity of process $\hat{Y}$, it holds $P$–a.s. that $Y_s^1 \leq (\text{resp.} \geq) Y_s^2$ for any $s \in [t,T]$. \hfill $\square$

**Proof of Lemma 2.1** It suffices to prove for $U_t$–controls. Let $s \in [t,T]$ and $U \in \mathcal{B}(\mathbb{U})$. Since $[t, \tau], [\tau, T] \in \mathcal{P}$, we see that both $\mathcal{D}_1 \overset{\triangle}{=} \{ [t, \tau] \cap \{(s, x) \times \Omega \}$ and $\mathcal{D}_2 \overset{\triangle}{=} \{ [\tau, T] \cap \{(s, x) \times \Omega \}$ belong to $\mathcal{B}(\{t, s\}) \otimes \mathcal{F}_s$. It then follows that

$$\{ (r, \omega) \in [t, s] \times \Omega : \mu_r^s(\omega) \in U \} = \{ (r, \omega) \in \mathcal{D}_1 : \mu_r^s(\omega) \in U \} \cup \{ (r, \omega) \in \mathcal{D}_2 : \mu_r^s(\omega) \in U \} = (\mathcal{D}_1 \cap \{ (r, \omega) \in [t, s] \times \Omega : \mu_r^s(\omega) \in U \}) \cup (\mathcal{D}_2 \cap \{ (r, \omega) \in [t, s] \times \Omega : \mu_r^s(\omega) \in U \}) \in \mathcal{B}(\{t, s\}) \otimes \mathcal{F}_s,$$

which shows that the process $\mu$ is $\hat{F}$–progressively measurable.

For $i = 1, 2$, suppose that $E \int_t^T |\mu_{s,i}^r|^{q_i} \, ds < \infty$ for some $q_i > 2$. One can deduce that $E \int_t^T |\mu_{s,1}^r|^{q_1 \wedge q_2} \, ds \leq E \int_t^T |\mu_{s,1}^r|^{q_1} \, ds + E \int_t^T |\mu_{s,2}^r|^{q_2} \, ds < \infty$. Thus $\mu \in \mathcal{U}_t$. \hfill $\square$

**Proof of Lemma 2.2** Both $\{ X_{s \wedge \Lambda} \} \in [t,T]$ and $\{ X_{s \wedge \Lambda}^t \} \in [t,T]$ satisfy the same SDE:

$$X_s = \xi + \int_t^s b_r^\mu (r, X_r) \, dr + \int_t^s \sigma_r^\mu (r, X_r) \, dB_r, \quad s \in [t,T], \tag{4.5}$$

where $b_r^\mu (r, x, \omega) \overset{\triangle}{=} 1_{(r < \tau(\omega))} b_{r, \omega}^\mu (r, \omega, x)$ and $\sigma_r^{\mu, r}(r, \omega, x, \omega) \overset{\triangle}{=} 1_{(r < \tau(\omega))} \sigma_{r, \omega}^{\mu, r} (r, \omega, x, \omega)$, $\forall (r, \omega, x) \in [t,T] \times \Omega \times \mathbb{R}^k$. Like $b_r^\mu$ and $\sigma_r^{\mu, r}$ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^k) / \mathcal{B}(\mathbb{R}^k)$–measurable function and $\sigma_r^{\mu, r}$ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^k) / \mathcal{B}(\mathbb{R}^k \times d)$–measurable function that is Lipschitz continuous in $(y, z)$ with coefficient $\gamma$ and satisfies

$$E \left[ \left( \int_t^T |b_r^\mu (s, 0)| \, ds \right)^2 + \left( \int_t^T |\sigma_r^{\mu, r} (s, 0)| \, ds \right)^2 \right] < \infty.$$
Thus (4.5) has a unique solution. It then holds $P$–a.s. that
\[ X^t_{\tau,A} = X^t_{\tau,A}, \quad \forall s \in [t, T]. \tag{4.6} \]

One can also deduce that
\[ X^t_{\tau,A} - X^t_{\tau,B} = \int_t^\tau b(r, X^r_{\tau,A}, \mu, \nu)dr + \int_t^\tau \sigma(r, X^r_{\tau,A}, \mu, \nu)dB_r, \quad s \in [t, T]. \]

Multiplying $1_A$ on both sides yields that
\[
X_s^A \triangleq 1_A(X^t_{\tau,A} - X^t_{\tau,B}) = \int_t^s 1_A b(r, X^r_{\tau,A}, \mu, \nu)dr + \int_t^s 1_A \sigma(r, X^r_{\tau,A}, \mu, \nu)dB_r
\]
\[ = \int_t^s 1_{\{r \geq \tau\}} 1_A b(r, X^r_{\tau,A} + X^r_{\tau,B}, \mu, \nu)dr + \int_t^s 1_{\{r \geq \tau\}} 1_A \sigma(r, X^r_{\tau,A} + X^r_{\tau,B}, \mu, \nu)dB_r, \quad s \in [t, T]. \]

Similarly, we see from (4.6) that
\[
X_s \triangleq 1_A(X^t_{\tau,A} - X^t_{\tau,B}) = \int_t^s 1_{\{r \geq \tau\}} 1_A b(r, X^r_{\tau,A}, \mu, \nu)dr + \int_t^s 1_{\{r \geq \tau\}} 1_A \sigma(r, X^r_{\tau,A}, \mu, \nu)dB_r
\]
\[ = \int_t^s 1_{\{r \geq \tau\}} 1_A b(r, X^r_{\tau,A} + X^r_{\tau,B}, \mu, \nu)dr + \int_t^s 1_{\{r \geq \tau\}} 1_A \sigma(r, X^r_{\tau,A} + X^r_{\tau,B}, \mu, \nu)dB_r, \quad s \in [t, T]. \]

To wit, $X, X \in \mathbb{C}^2_p([t, T], \mathbb{R}^k)$ satisfy the same SDE:
\[ X_s = \int_t^s \tilde{b}(r, X_r)dr + \int_t^s \tilde{\sigma}(r, X_r)dB_r, \quad s \in [t, T], \tag{4.7} \]

where $\tilde{b}(r, x, \omega) \triangleq 1_{\{r \geq \tau(x)\}} 1_{\{\omega \in A\}} b(r, x, X^r_{\tau,A}(\omega), \mu(\omega), \nu(\omega))$ and $\tilde{\sigma}(r, x, \omega) \triangleq 1_{\{r \geq \tau(x)\}} 1_{\{\omega \in A\}} \sigma_{\mu}(r, x, X^r_{\tau,A}(\omega), \mu(\omega), \nu(\omega)), \forall (r, x, \omega) \in [t, T] \times \mathbb{R} \times \mathbb{R}^k$. The measurability of functions $b, X^r_{\tau,A}, \mu$ and $\nu$ implies that the mapping $(r, x, \omega) \to b(r, x, X^r_{\tau,A}(\omega), \mu(\omega), \nu(\omega))$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^k)/\mathcal{B}(\mathbb{R}^k)$–measurable. Clearly, $(1_{\{r \geq \tau(x)\}} 1_{\{\omega \in A\}})_{r \in [t, T]}$ is a right-continuous $\mathcal{F}$–adapted process. Thus $\tilde{b}$ is also $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^k)/\mathcal{B}(\mathbb{R}^k)$–measurable. Similarly, $\tilde{\sigma}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^k)/\mathcal{B}(\mathbb{R}^{k \times d})$–measurable. By (2.2), both $\tilde{b}$ and $\tilde{\sigma}$ are Lipschitz continuous in $x$. Since
\[
E \left[ \left( \int_t^T \hat{b}(r, 0) \right)^2 + \left( \int_t^T \hat{\sigma}(r, 0) \right)^2 \right] \leq c_0 + c_0E \left[ \left( X^r_{\tau,A} \right)^2 \right] < \infty
\]
by (2.1), (2.2) and Hölder’s inequality, the SDE (4.7) admits a unique solution. Hence, $P(X_s = X_s, \forall s \in [t, T]) = 1$, which together with (4.6) proves (2.7).

**Proof of Lemma 2.3.** For $i = 1, 2$, let $\Theta_i \triangleq (t, \xi_i, \mu, \nu)$ and set $(Y^i, Z^i) \triangleq \left(Y^{\Theta_i}(T, g(X^\Theta_i)), Z^{\Theta_i}(T, g(X^\Theta_i))\right)$. Given $\tilde{p} \in (1, p]$, (2.4) and Hölder’s inequality show that
\[
E \left[ \left( \int_t^T |f^\Theta_1(r, Y^1_r, Z^1_r) - f^\Theta_2(r, Y^2_r, Z^2_r)|ds \right)^{\tilde{p}} \right] \leq c_p E \left[ \sup_{s \in [t, T]} |X^{\Theta_1}_s - X^{\Theta_2}_s|^2 \right]^{\tilde{p}/2} < \infty.
\]

Then we can deduce from (1.5) that
\[
E \left[ \sup_{s \in [t, T]} |Y^1_s - Y^2_s|^2 \right] \leq c_p E \left[ \left| g(X^{\Theta_1}) - g(X^{\Theta_2}) \right|^2 + \int_t^T |f^\Theta_1(r, Y^1_r, Z^1_r) - f^\Theta_2(r, Y^2_r, Z^2_r)|^2ds \right] \leq c_p E \left[ \sup_{s \in [t, T]} |X^{\Theta_1}_s - X^{\Theta_2}_s|^2 \right]^{\tilde{p}/2}, \quad P-a.s.
\]

Then a standard a priori estimate of SDEs (see e.g. [19] pg. 166-168 and [20] pg. 289-290) leads to that
\[
E \left[ \sup_{s \in [t, T]} |Y^1_s - Y^2_s|^2 \right] \leq c_p E \left[ \sup_{s \in [t, T]} |X^{\Theta_1}_s - X^{\Theta_2}_s|^2 \right]^{\tilde{p}/2}, \quad P-a.s.
\]
Proof of Proposition 2.1: Given $\beta \in \mathcal{B}_t$, (1.4) and Hölder’s inequality imply that
\[
\left| J(t, x, u_0, \beta(u_0)) \right|^p \leq E \left[ \sup_{s \in [t, T]} \left| Y^{t, x, u_0, \beta(u_0)}(T, g(X_T^{t, x, u_0, \beta(u_0)})) \right|^p \right] 
\leq c_0 E \left[ g(X_T^{t, x, u_0, \beta(u_0)})^p + \int_t^T \left| f_T^{t, x, u_0, \beta(u_0)}(s, 0, 0) \right|^p ds \right] F, \quad P\text{-a.s.} \tag{4.8}
\]

Since $\left[ \left( \beta(u_0) \right) \right]_{s} \leq \kappa, \ ds \times dP\text{-a.s.}$, the $2/p$–Hölder continuity of $g$, (2.3), (2.4) as well as a conditional-expectation version of (2.5) show that $P$–a.s.
\[
| J(t, x, u_0, \beta(u_0)) |^p \leq c_0 + c_0 E \left[ X_T^{t, x, u_0, \beta(u_0)}^2 + \int_t^T \left( |X_T^{t, x, u_0, \beta(u_0)}|^2 + \left[ (\beta(u_0)) \right]_{s}^2 \right) ds \right] F \tag{4.9}
\]

So it follows that
\[
w_1(t, x) \geq \text{essinf}_{\beta \in \mathcal{B}_t} J(t, x, u_0, \beta(u_0)) \geq -c_\kappa - c_0|x|^{2/p}, \quad P\text{-a.s.}
\]

We extensively set $\psi(u) \overset{\triangle}{=} v_0, \ \forall (t, u) \in [0, T] \times O_k(u_0)$, then it is $\mathcal{B}([0, T]) \times \mathcal{B}(U) / \mathcal{B}(V)$ measurable. For any $\mu \in \mathcal{U}_t$, the measurability of function $\psi$ and process $\mu$ implies that
\[
(\beta_\psi(\mu))_s \overset{\triangle}{=} \psi(s, \mu_s), \quad s \in [t, T] \tag{4.10}
\]
defines a $\mathcal{V}$–valued, $\mathcal{F}$–progressively measurable process, and we see from (A-u) that $\left[ (\beta_\psi(\mu)) \right]_{s} \leq \kappa + \kappa|\mu_s|_U$, \ \forall s \in [t, T]. So $\beta_\psi(\mu) \in \mathcal{V}_t$. Let $\mu^1, \mu^2 \in \mathcal{U}_t$ such that $\mu^1 = \mu^2, \ ds \times dP$–a.s. on $[t, \tau] \cup [\tau, T]_A$ for some $\tau \in \mathcal{S}_{t,T}$ and $A \in \mathcal{F}_\tau$. It clearly holds $ds \times dP$–a.s. on $[t, \tau] \cup [\tau, T]_A$ that
\[
(\beta_\psi(\mu^1))_s = \psi(s, \mu^1_s) = \psi(s, \mu^2_s) = (\beta_\psi(\mu^2))_s.
\]

Hence, $\beta_\psi \in \mathcal{B}_t$.

Fix a $u_2 \in \partial O_k(u_0)$. For any $\mu \in \mathcal{U}_t$, similar to (4.8) and (4.9), we can deduce that $P$–a.s.
\[
| J(t, x, \mu, \beta_\psi(\mu)) |^p \leq c_0 E \left[ \left| g(X_T^{t, x, \mu, \beta_\psi(\mu)}) \right|^p + \int_t^T \left| f_T^{t, x, \mu, \beta_\psi(\mu)}(s, 0, 0) \right|^p ds \right] F \tag{4.11}
\]

where we used a conditional-expectation version of (2.5) in the last inequality. Then an analogous decomposition and estimation to (4.11) leads to that $| J(t, x, \mu, \beta_\psi(\mu)) |^p \leq c_\kappa + c_0|x|^2$, $P$–a.s. It follows that
\[
w_1(t, x) \leq \text{esssup}_{\mu \in \mathcal{U}_t} J(t, x, \mu, \beta_\psi(\mu)) \leq c_\kappa + c_0|x|^{2/p}, \quad P\text{-a.s.}
\]

Similarly, one has $|w_2(t, x)| \leq c_\kappa + c_0|x|^{2/p}, \quad P\text{-a.s.}$

Proof of Proposition 2.2: Let $\mathcal{H}$ denote the Cameron-Martin space of all absolutely continuous functions $h \in \Omega$ whose derivative $\dot{h}$ belongs to $L^2([0, T], \mathbb{R}^d)$. For any $h \in \mathcal{H}$, we define $\mathcal{T}_h(\omega) \overset{\triangle}{=} \omega + h$, $\forall \omega \in \Omega$. Clearly, $\mathcal{T}_h : \Omega \to \Omega$
is a bijection and its law is given by $P_h \overset{\Delta}{=} P \circ T_h^{-1} = \exp \left\{ \int_0^T \dot{h}_s dB_s - \frac{1}{2} \int_0^T |\dot{h}_s|^2 ds \right\} P$. Fix $(t,x) \in [0,T] \times \mathbb{R}^k$ and set $\mathcal{H}_t \overset{\Delta}{=} \{ h \in \mathcal{H} : h(s) = h(s \wedge t), \; \forall s \in [0,T] \}$.

Fix $h \in \mathcal{H}_t$. We first show that

$$ (\mu(T_h), \nu(T_h)) \in \mathcal{U}_t \times \mathcal{V}_t, \quad \forall (\mu, \nu) \in \mathcal{U}_t \times \mathcal{V}_t. $$

(4.12)

Let $\mu \in \mathcal{U}_t$. Given $s \in [t,T]$, we set $\mathcal{Y}_s^h(D) \overset{\Delta}{=} \{ (r, \omega) \in [t,s] \times \Omega : (r, T_h(\omega)) \in D \}$ for any $D \subset [t,s] \times \Omega$. As the mapping

$$ T_h = B + h \text{ is } \mathcal{F}_s / \mathcal{F}_s - \text{measurable}, $$

(4.13)

it holds for any $\mathcal{E} \in \mathcal{B}([t,s])$ and $A \in \mathcal{F}_s$ that

$$ \mathcal{Y}_s^h(\mathcal{E} \times A) = \{ (r, \omega) \in [t,s] \times \Omega : (r, T_h(\omega)) \in \mathcal{E} \times A \} = (\mathcal{E} \cap [t,s]) \times T_h^{-1}(A) \in \mathcal{B}([t,s]) \otimes \mathcal{F}_s, $$

So $\mathcal{E} \times A \in \Lambda^h \overset{\Delta}{=} \{ D \subset [t,s] \times \Omega : \mathcal{Y}_s^h(D) \in \mathcal{B}([t,s]) \otimes \mathcal{F}_s \}$. In particular, $\emptyset \times \emptyset \in \Lambda^h$ and $[t,s] \times \Omega \in \Lambda^h$. For any $D \in \Lambda^h$ and $\{ D_n \}_{n \in \mathbb{N}} \subset \Lambda^h$, one can deduce that

$$ \mathcal{Y}_s^h \left( \bigcup_{n \in \mathbb{N}} D_n \right) = \{ (r, \omega) \in [t,s] \times \Omega : (r, T_h(\omega)) \in \bigcup_{n \in \mathbb{N}} D_n \} = \bigcup_{n \in \mathbb{N}} \mathcal{Y}_s^h(D_n) \in \mathcal{B}([t,s]) \otimes \mathcal{F}_s, $$

i.e. $([t,s] \times \Omega) \setminus \bigcup_{n \in \mathbb{N}} D_n \in \Lambda^h$. Thus $\Lambda^h$ is a $\sigma$–field of $[t,s] \times \Omega$. It follows that

$$ \mathcal{B}([t,s]) \otimes \mathcal{F}_s = \sigma \{ \mathcal{E} \times A : \mathcal{E} \in \mathcal{B}([t,s]), \; A \in \mathcal{F}_s \} \subset \Lambda^h. $$

(4.14)

Given $U \in \mathcal{B}(U)$, the $\mathbf{F}$–progressive measurability of $\mu$ and (4.14) show that

$$ \mathcal{D}_U \overset{\Delta}{=} \{ (r, \omega) \in [t,s] \times \Omega : \mu_r(T_h(\omega)) \in U \} \in \mathcal{B}([t,s]) \otimes \mathcal{F}_s \subset \Lambda^h. $$

That is

$$ \{ (r, \omega) \in [t,s] \times \Omega : \mu_r(T_h(\omega)) \in U \} = \{ (r, \omega) \in [t,s] \times \Omega : (r, T_h(\omega)) \in \mathcal{D}_U \} = \mathcal{Y}_s^h(\mathcal{D}_U) \in \mathcal{B}([t,s]) \otimes \mathcal{F}_s, $$

(4.15)

which shows that the $\mathbf{F}$–progressive measurability of process $\mu(T_h)$.

Suppose that $E \int_t^T |\mu_s(\nabla)|^q d\mathbf{s} < \infty$ for some $q > 2$. Then one can deduce that for any $\bar{q} \in (2, q)$

$$ E \int_t^T |\mu_s(T_h)|^\bar{q} d\mathbf{s} = E_{\mu_h} \int_t^T |\mu_s|^q d\mathbf{s} = E \left[ \exp \left\{ \int_0^T \dot{h}_s dB_s - \frac{1}{2} \int_0^T |\dot{h}_s|^2 ds \right\} \int_t^T |\mu_s|^q d\mathbf{s} \right] \\ \leq T^{2,2} \exp \left( \frac{\bar{q}}{2(q - \bar{q})} \int_0^T |\dot{h}_s|^2 ds \right) E \left[ \exp \left\{ \int_0^T \dot{h}_s dB_s - \frac{\bar{q}}{2(q - \bar{q})} \int_0^T |\dot{h}_s|^2 ds \right\} \left( \int_t^T |\mu_s|^\bar{q} d\mathbf{s} \right) \right] \\ \leq T^{2,2} \exp \left( \frac{\bar{q}}{2(q - \bar{q})} \int_0^T |\dot{h}_s|^2 ds \right) \left( E \left[ \exp \left\{ \frac{\bar{q}}{q - \bar{q}} \int_0^T \dot{h}_s dB_s - \frac{\bar{q}^2}{2(q - \bar{q})} \int_0^T |\dot{h}_s|^2 ds \right\} \right] \right)^{\frac{q - \bar{q}}{q - \bar{q}}} < \infty. $$

(4.16)

Hence, $\mu(T_h) \in \mathcal{U}_t$. Similarly, $\nu(T_h) \in \mathcal{V}_t$ for any $\nu \in \mathcal{V}_t$.

Let $\{ \Phi_s \}_{s \in [t,T]}$ be an $\mathbb{R}^{k \times d}$–valued, $\mathbf{F}$–progressively measurable process and set $M_s \overset{\Delta}{=} \int_s^T \Phi_r dB_r$, $s \in [t,T]$. We know that (see e.g. Problem 3.2.27 of [20]) there exists a sequence of $\mathbb{R}^{k \times d}$–valued, $\mathbf{F}$–simple processes
\[
\left\{ \Phi^n_s = \sum_{i=1}^{\ell_n} \xi^n_i \mathbb{1}_{\{s \in (t^n_i, t^n_{i+1})\}}, s \in [t, T] \right\}_{n \in \mathbb{N}} \quad \text{(where } t = t^n_1 < \cdots < t^n_{\ell_n+1} = T \text{ and } \xi^n_i \in \mathcal{F}_{t^n_i} \text{ for } i = 1, \cdots, \ell_n) \text{ such that }
\]
\[
P - \lim_{n \to \infty} \int_t^T \text{tr}\left\{ \left( \Phi^n_r - \Phi^s_r \right) \left( \Phi^n_r - \Phi^s_r \right)^T \right\} ds = 0 \quad \text{and} \quad P - \lim_{n \to \infty} \sup_{s \in [t, T]} |M^n_s - M_s| = 0,
\]
where \(M^n_s \overset{d}{=} \int_t^s \Phi^n_r dB_r\). By the equivalence of \(P_h\) to \(P\), one has
\[
P_h - \lim_{n \to \infty} \int_t^T \text{tr}\left\{ \left( \Phi^n_r - \Phi^s_r \right) \left( \Phi^n_r - \Phi^s_r \right)^T \right\} ds = P_h - \lim_{n \to \infty} \sup_{s \in [t, T]} |M^n_s - M_s| = 0,
\]
or
\[
P - \lim_{n \to \infty} \int_t^T \text{tr}\left\{ \left( \Phi^n_r(\Theta_h) - \Phi^s_r(\Theta_h) \right) \left( \Phi^n_r(\Theta_h) - \Phi^s_r(\Theta_h) \right)^T \right\} ds = P - \lim_{n \to \infty} \sup_{s \in [t, T]} |M^n_s(\Theta_h) - M_s(\Theta_h)| = 0. \quad (4.17)
\]
Applying Proposition 3.2.26 of [20] yields that
\[
0 = P - \lim_{n \to \infty} \sup_{s \in [t, T]} \left| \int_t^s \Phi^n_r(\Theta_h)dB_r - \int_t^s \Phi_r(\Theta_h)dB_r \right|. \quad (4.18)
\]
As \(h \in \mathcal{H}_t\), one can deduce that
\[
M^n_s(\Theta_h) = \left( \sum_{i=1}^{\ell_n} \xi^n_i \left( B_{s \land t^n_{i+1}} - B_{s \land t^n_i} \right) \right)(\Theta_h) = \sum_{i=1}^{\ell_n} \xi^n_i(\Theta_h) \left( B_{s \land t^n_{i+1}}(\Theta_h) - B_{s \land t^n_i}(\Theta_h) \right)
\]
\[
= \sum_{i=1}^{\ell_n} \xi^n_i(\Theta_h) \left( B_{s \land t^n_{i+1}} - h(s \land t^n_{i+1}) - B_{s \land t^n_i} + h(s \land t^n_i) \right) = \int_t^s \Phi^n_r(\Theta_h)dB_r, \quad \forall s \in [t, T],
\]
which together with (4.17) and (4.18) leads to that \(P-a.s.\)
\[
\int_t^s \Phi_r(\Theta_h)dB_r = M_s(\Theta_h) = \left( \int_t^s \Phi_rdB_r \right)(\Theta_h), \quad s \in [t, T]. \quad (4.19)
\]
Let \((\mu, \nu) \in \mathcal{U}_t \times \mathcal{V}_t\) and set \(\Theta = (t, x, \mu, \nu)\). By (4.13), the process \(X^{\Theta}(\Theta_h)\) is \(\mathbb{F}\)-adapted, and the equivalence of \(P_h\) to \(P\) implies that \(X^{\Theta}(\Theta_h)\) has \(P-a.s.\) continuous paths. Suppose that \(Ef^T_{t} [\mu_s]_q^q ds + Ef^T_{t} [\nu_s]_q^q ds < \infty\) for some \(q > 2\). A standard estimate of SDEs (see e.g. [19] pg. 166-168 and [20] pg. 289-290) shows that
\[
E \left[ \sup_{s \in [t, T]} \left| X^{\Theta}_s(\Theta_h) \right|^q \right] \leq c_q |x|^q + c_q E \left[ \int_t^T |b^{\mu, \nu}(s, 0)| ds \right]^q + \left( \int_t^T |\sigma^{\mu, \nu}(s, 0)| ds \right)^q \leq c_q \left( 1 + |x|^q + E \int_t^T [\mu_s]_q^q + [\nu_s]_q^q ds \right) < \infty. \quad (4.20)
\]
Similar to (4.16), one can deduce that \(E \left[ \sup_{s \in [t, T]} \left| X^{\Theta}_s(\Theta_h) \right|^q \right] < \infty\) for any \(\tilde{q} \in [2, q)\). In particular, \(X^{\Theta}(\Theta_h) \in \mathbb{C}^2_{\mathbb{F}}([t, T], \mathbb{R}^k)\). It follows from (4.19) that
\[
X^{\Theta}_s(\Theta_h) = x + \int_t^s b(r, X^{\Theta}_r(\Theta_h), \mu_r(\Theta_h), \nu_r(\Theta_h)) dr + \left( \int_t^s \sigma(r, X^{\Theta}_r(\Theta_h), \mu_r(\Theta_h), \nu_r(\Theta_h)) dB_r \right)(\Theta_h)
\]
\[
= x + \int_t^s b(r, X^{\Theta}_r(\Theta_h), \mu_r(\Theta_h), \nu_r(\Theta_h)) dr + \int_t^s \sigma(r, X^{\Theta}_r(\Theta_h), \mu_r(\Theta_h), \nu_r(\Theta_h)) dB_r, \quad s \in [t, T].
\]
Thus the uniqueness of SDE (1.1) with parameters \(\Theta_h = (t, x, \mu, \nu)\) shows that
\[
X^{\Theta}_s(\Theta_h) = X^{\Theta}_s(\Theta_h), \quad \forall s \in [t, T]. \quad (4.21)
\]
Let \((\hat{Y}, \hat{Z}) = (Y^{\Theta}(T, g(X^{\Theta}_T)), Z^{\Theta}(T, g(X^{\Theta}_T)))\). Analogous to \(X^{\Theta}(\Theta_h), \hat{Y}(\Theta_h)\) is an \(\mathbb{F}\)-adapted continuous process. And using the similar arguments that leads to (4.15), we see that the process \(\hat{Z}(\Theta_h)\) is \(\mathbb{F}\)-progressively measurable. By (4.20), \(g(X^{\Theta}_T) \in \mathbb{L}^{2, T}_{\mathbb{F}}(\mathcal{F}_T)\), and a similar argument to (2.8) yields that
\[
E \left[ \left( \int_t^T |f^{\mu, \nu}_r(s, 0, 0)| ds \right)^{\tilde{q}} \right] \leq c_q + c_q E \left[ \sup_{s \in [t, T]} \left| X^{\Theta}_s \right|^{\tilde{q}} \right] + \int_t^T [\mu_s]_q^q + [\nu_s]_q^q ds < \infty.
\]
Then we know from Proposition 1.1 that the unique solution \((\bar{Y}, \bar{Z})\) of BSDE\(_t\) \((t, g(X^\Theta_t), f^\Theta_t)\) in \(\mathbb{D}_T^2([t, T])\) actually belongs to \(\mathbb{G}_T^\Theta([t, T])\). Similar to (4.16), one can deduce that \(E\left[\sup_{s \in [t,T]} |\bar{Y}_s(T_h)|^{\tilde{q}} + \left(\int_0^T |\bar{Z}_s(T_h)|^2 ds\right)^{q/2}\right] < \infty\) for any \(\tilde{q} \in [p, \frac{2}{p'}]\). In particular, \((\bar{Y}(T_h), \bar{Z}(T_h)) \in \mathbb{G}_T^\Theta([t, T])\).

Applying (4.19) again, we can deduce from (4.21) that
\[
\hat{Y}_s(T_h) = g(X^\Theta_T(T_h)) + \int_s^T f(r, X^\Theta_r(T_h), \bar{Y}_r(T_h), \bar{Z}_r(T_h), \mu_r(T_h), \nu_r(T_h)) dr - \left(\int_s^T \bar{Z}_r dB_r\right)(T_h)
\]
\[
= g(X^\Theta_{T_h}) + \int_s^T f(r, X^\Theta_r, \bar{Y}_r(T_h), \bar{Z}_r(T_h), \mu_r(T_h), \nu_r(T_h)) dr - \int_s^T \bar{Z}_r dB_r, \quad s \in [t, T].
\]
Thus the uniqueness of BSDE\(_t\) \((t, g(X^\Theta_T), f^\Theta_T)\) implies that \(P\)-a.s.
\[
Y_{s}^{\Theta_h}(T, g(X^\Theta_T)) = \hat{Y}_s(T_h), \quad s \in [t, T].
\]

In particular,
\[
J(t, x, \mu, \nu)(T_h) = \hat{Y}_t(T_h) = Y_{t}^{\Theta_h}(T, g(X^\Theta_T)) = J(t, x, \mu(T_h), \nu(T_h)), \quad P\text{-a.s.} \tag{4.22}
\]

Next, let \(\beta \in \mathcal{B}_1\) and define
\[
\beta_h(\mu) \triangleq \beta(\mu(T-h)) (T_h), \quad \forall \mu \in \mathcal{U}_t.
\]

similar to (1.12), \(\mu(T-h) \in \mathcal{U}_t\) as \(-h\) also belongs to \(\mathcal{H}\). It follows that \(\beta(\mu(T-h)) \in \mathcal{V}_t\). Using (1.12) again shows that \(\beta_h(\mu) = \beta(\mu(T-h))(T_h) \in \mathcal{V}_t\). Since \([\beta(\mu(T-h))](\mu) \leq \kappa + C_\beta h(T-h)\), \(ds \times dP\)-a.s., the equivalence of \(P_h\) to \(P\) shows that \([\beta(\mu(T-h))](\mu) \leq \kappa + C_\beta h(T-h)\), \(ds \times dP_h\)-a.s., or
\[
[\beta(\mu(T-h))](\mu) \leq [\beta(\mu(T-h))](\mu) \leq \kappa + C_\beta h(T-h)\), \(ds \times dP\)-a.s.
\]

Let \(\mu_1,\mu_2 \in \mathcal{U}_t\) such that \(\mu_1 = \mu_2\), \(ds \times dP\)-a.s. on \([t, \tau] \cup [\tau, T]_A\) for some \(\tau \in S_{t,T}\) and \(A \in \mathcal{F}_t\). By the equivalence of \(P_h\) to \(P\), \(\mu_1 = \mu_2\), \(ds \times dP_h\)-a.s. on \([t, \tau] \cup [\tau, T]_A\), or \(\mu_1(T-h) = \mu_2(T-h)\), \(ds \times dP\)-a.s. on \([\tau\tau] \cup [\tau(T-h), T]_{T_h(A)}\). Given \(s \in [t, T]\), similar to (4.13), \(T_h\) is also \(\mathcal{F}_s/\mathcal{F}_s\)-measurable. It follows that
\[
\{\tau(T-h) \leq s\} = \{\omega: \omega(T-h)(\omega) \in \{\tau \leq s\}\} = T^{-1}_h(\{\tau \leq s\}) \in \mathcal{F}_s
\]
and \((T_h(A) \cap \{\tau(T-h) \leq s\}) = T^{-1}_h(A \cap \{\tau \leq s\}) = T^{-1}_h(A \cap \{\tau \leq s\}) \in \mathcal{F}_s\),

which shows that \(\tau(T-h)\) is an \(\mathcal{F}\)-stopping time and \(T_h(A) \in \mathcal{F}_{\tau(T-h)}\). As \(t \leq \tau \leq T\), \(P\text{-a.s.}\), the equivalence of \(P_{h}\) to \(P\) shows that \(t \leq \tau \leq T\), \(P_{h}\)-a.s., or \(t \leq \tau(T-h) \leq T\), \(P\text{-a.s.}\). So \(\tau(T-h) \in \mathcal{S}_{t,T}\), and we see from Definition 2.2 that \(\beta(\mu_1(T-h)) = \beta(\mu_2(T-h))\), \(ds \times dP\)-a.s. on \([t, \tau(T-h)] \cup [\tau(T-h), T]_{T_h(A)}\). The equivalence of \(P_h\) to \(P\) then shows that \(\beta(\mu_1(T-h)) = \beta(\mu_2(T-h))\), \(ds \times dP_h\)-a.s. on \([t, \tau(T-h)] \cup [\tau(T-h), T]_{T_h(A)}\), or \(\beta(\mu_1(T-h)) = \beta(\mu_2(T-h))\)(\(T_h\)) = \(\beta(\mu_2(T-h))\)(\(T_h\)) = \(\beta(\mu_2(T-h))\), \(ds \times dP\)-a.s. on \([t, \tau(T-h)] \cup [\tau(T-h), T]_{T_h(A)}\). Hence, \(\beta_h \in \mathcal{B}_t\).

Set \(I(t, x, \beta) \triangleq \operatorname{esssup}_{\mu \in \mathcal{U}_t} J(t, x, \mu, \beta(\mu))\). For any \(\mu \in \mathcal{U}_t\), as \(I(t, x, \beta) \geq J(t, x, \mu, \beta(\mu))\), \(P\text{-a.s.}\), the equivalence of \(P_h\) to \(P\) shows that \(I(t, x, \beta) \geq J(t, x, \mu, \beta(\mu))\), \(P_h\)-a.s., or
\[
I(t, x, \beta)(T_h) \geq J(t, x, \mu, \beta(\mu))(T_h), \quad P\text{-a.s.} \tag{4.23}
\]

Let \(\xi\) be another random variable such that \(\xi \geq J(t, x, \mu, \beta(\mu))(T_h)\), \(P\text{-a.s.}\), or \(\xi(T-h) \geq J(t, x, \mu, \beta(\mu))\), \(P_h\text{-a.s.}\) for any \(\mu \in \mathcal{U}_t\). By the equivalence of \(P_h\) to \(P\), it holds for any \(\mu \in \mathcal{U}_t\) that \(\xi(T-h) \geq J(t, x, \mu, \beta(\mu))\), \(P\text{-a.s.}\). Taking essential supremum over \(\mu \in \mathcal{U}_t\) yields that \(\xi(T-h) \geq I(t, x, \beta)\), \(P\text{-a.s.}\) or \(\xi \geq I(t, x, \beta)(T_h)\), \(P_h\text{-a.s.}\). Then it follows from the equivalence of \(P_{h}\) to \(P\) that \(\xi \geq I(t, x, \beta)(T_h)\), \(P\text{-a.s.}\), which together with (4.23) implies that
\[
\operatorname{esssup}_{\mu \in \mathcal{U}_t} \left(\left.I(t, x, \mu, \beta(\mu))(T_h)\right) = I(t, x, \beta)(T_h) = \left(\operatorname{esssup}_{\mu \in \mathcal{U}_t} J(t, x, \mu, \beta(\mu))\right)(T_h), \quad P\text{-a.s.} \tag{4.24}
\]
Similarly, \( \text{essinf}_{\beta \in \mathbb{B}_t} \left( I(t, x, \beta)(T_h) \right) = \left( \text{essinf}_{\beta \in \mathbb{B}_t} I(t, x, \beta) \right)(T_h) \), \( P \)-a.s., which together (4.22) and (4.24) yields that
\[
w_1(t, x)(T_h) = \left( \text{essinf}_{\beta \in \mathbb{B}_t} I(t, x, \beta)(T_h) \right) = \text{essinf}_{\beta \in \mathbb{B}_t} (J(t, x, \mu, \beta(\mu))(T_h)) = \text{essinf}_{\beta \in \mathbb{B}_t, \mu \in \mathcal{U}_t} J(t, x, \mu, \beta(\mu)) = \text{essinf}_{\beta \in \mathbb{B}_t, \mu \in \mathcal{U}_t} J(t, x, \mu, \beta(\mu)) = w_1(t, x), \quad P \text{-a.s.} \tag{4.25}
\]
where we used the facts that \( \{\mu(T_h) : \mu \in \mathcal{U}_t\} = \mathcal{U}_t \) and \( \{\beta : \beta \in \mathbb{B}_t\} = \mathbb{B}_t \).

As an \( \mathcal{F}_t \)-measurable random variable, \( w_1(t, x) \) only depends on the restriction of \( \omega \in \Omega \) to the time interval \( [0, t] \). So (4.25) holds even for any \( h \in \mathcal{H} \). Then an application of Lemma 3.4 of [10] yields that \( w_1(t, x) = E[w_1(t, x)], \) \( P \)-a.s. Similarly, one can deduce that \( w_2(t, x) = E[w_2(t, x)], \) \( P \)-a.s.

**Proof of Proposition 2.3**

Let \( t \in [0, T] \) and \( x_1, x_2 \in \mathbb{R}^k \). For any \((\beta, \mu) \in \mathbb{B}_t \times \mathcal{U}_t \), (2.13) implies that
\[
|J(t, x_1, \mu, \beta(\mu)) - J(t, x_2, \mu, \beta(\mu))|^p \leq c_0|x_1 - x_2|^2, \quad P \text{-a.s.}
\]
which leads to that
\[
J(t, x_2, \mu, \beta(\mu)) - c_0|x_1 - x_2|^2/p \leq J(t, x_1, \mu, \beta(\mu)) \leq J(t, x_2, \mu, \beta(\mu)) + c_0|x_1 - x_2|^2/p, \quad P \text{-a.s.}
\]
Taking essential supremum over \( \mu \in \mathcal{U}_t \) and then taking essential infimum over \( \beta \in \mathbb{B}_t \) yield that
\[
w_1(t, x_2) - c_0|x_1 - x_2|^2/p \leq w_1(t, x_1) \leq w_1(t, x_2) + c_0|x_1 - x_2|^2/p.
\]
So \( |w_1(t, x_1) - w_1(t, x_2)| \leq c_0|x_1 - x_2|^2/p \). Similarly, one has \( |w_2(t, x_1) - w_2(t, x_2)| \leq c_0|x_1 - x_2|^2/p \).

**Lemma 4.1.**

Given \( t \in [0, T] \), let \( \{A_i\}_{i=1}^n \subset \mathcal{F}_t \) be a partition of \( \Omega \). For any \( \{(\xi_i, \mu_i, \nu_i)\}_{i=0}^n \subset L^2(\mathcal{F}_t, \mathbb{R}^k) \times \mathcal{U}_t \times \mathcal{V}_t \), if \( \xi_0 = \sum_{i=1}^n 1_{A_i} \xi_i, \) \( P \)-a.s. and if \( (\mu_0, \nu_0) = \left( \sum_{i=1}^n 1_{A_i} \mu_i, \sum_{i=1}^n 1_{A_i} \nu_i \right) \), \( ds \times dP \)-a.s., then it holds \( P \)-a.s. that
\[
X^t_{s, \xi_0, \mu_0, \nu_0} = \sum_{i=1}^n 1_{A_i} X^t_{s, \xi_i, \mu_i, \nu_i}, \quad \forall s \in [t, T]. \tag{4.26}
\]
Moreover, for any \( \{(\tau_i, \eta_i)\}_{i=1}^n \subset S_{t,T} \times L^p(\mathcal{F}_T) \) such that each \( \eta_i \) is \( \mathcal{F}_t \)-measurable, if \( \tau_0 = \sum_{i=1}^n 1_{A_i} \tau_i, \) \( P \)-a.s. and if \( \eta_0 = \sum_{i=1}^n 1_{A_i} \eta_i, \) \( P \)-a.s., then it holds \( P \)-a.s. that
\[
Y^t_{s, \xi_0, \mu_0, \nu_0}(\tau_0, \eta_0) = \sum_{i=1}^n 1_{A_i} Y^t_{s, \xi_i, \mu_i, \nu_i}(\tau_i, \eta_i), \quad \forall s \in [t, T]. \tag{4.27}
\]
In particular, one has
\[
J(t, \xi_0, \mu_0, \nu_0) = \sum_{i=1}^n 1_{A_i} J(t, \xi_i, \mu_i, \nu_i), \quad P \text{-a.s.} \tag{4.28}
\]

**Proof:** Let \((X^i, Y^i, Z^i) = (X^{t, \xi_i, \mu_i, \nu_i}, Y^{t, \xi_i, \mu_i, \nu_i}(\tau_i, \eta_i), Z^{t, \xi_i, \mu_i, \nu_i}(\tau_i, \eta_i)) \) for \( i = 0, \cdots, n \). We define
\[
\mathcal{X}, \mathcal{Y}, \mathcal{Z} \triangleq \sum_{i=1}^n 1_{A_i} \mathcal{X}^i, \mathcal{Y}^i, \mathcal{Z}^i \subset C^2_F([t, T], \mathbb{R}^k) \times C^2_F([t, T]).
\]
For any \( s \in [t, T] \) and \( i = 1, \cdots, n \), multiplying \( 1_{A_i} \) to SDE (1.1) with parameters \((t, \xi_i, \mu_i, \nu_i)\), we can deduce that
\[
1_{A_i} X^i_s = 1_{A_i} \xi_i + 1_{A_i} \int_t^s b(r, X^i_r, \mu_r, \nu_r) \, dr + 1_{A_i} \int_t^s X^i_r, \mu_r, \nu_r) \, dB_r
\]
\[
= 1_{A_i} \xi_i + \int_t^s 1_{A_i} b(r, X^i_r, \mu_r, \nu_r) \, dr + \int_t^s 1_{A_i} \sigma(r, X^i_r, \mu_r, \nu_r) \, d\mathcal{B}_r
\]
\[
= 1_{A_i} \xi_i + \int_t^s 1_{A_i} b(r, X^i_r, \mu_r, \nu_r) \, dr + \int_t^s 1_{A_i} \sigma(r, X^i_r, \mu_r, \nu_r) \, d\mathcal{B}_r
\]
\[
= 1_{A_i} \xi_i + 1_{A_i} \int_t^s b(r, X^i_r, \mu_r, \nu_r) \, dr + 1_{A_i} \int_t^s \sigma(r, X^i_r, \mu_r, \nu_r) \, d\mathcal{B}_r, \quad P \text{-a.s.} \tag{4.29}
\]
Adding them up over $i \in \{1, \cdots, n\}$ and using the continuity of process $\mathcal{X}$ show that $P$-a.s.

$$\mathcal{X}_s = \xi_0 + \int_t^s b(r, \mathcal{X}_r, \mu_r^0, \nu_r^0) \, dr + \int_t^s \sigma(r, \mathcal{X}_r, \mu_r^0, \nu_r^0) \, dB_r, \quad s \in [t, T].$$

So $\mathcal{X} = X^{t, \xi_0, \mu^0, \nu^0}$, i.e., (4.26).

Next, for any $s \in [t, T]$ and $i = 1, \cdots, n$, similar to (4.29), multiplying $1_A_i$ to BSDE $\left( t, \eta_i, f_t^{r, \xi_i, \mu_i, \nu_i} \right)$ yields that

$$1_A_i Y_s^i = 1_A_i \eta_i + 1_A_i \int_t^T 1_{\{r < \tau_i\}} f(r, X_r^i, Y_r^i, Z_r^i, \mu_r^0, \nu_r^0) \, dr - 1_A_i \int_t^T Z_r^i \, dB_r$$

$$= 1_A_i \eta_i + 1_A_i \int_t^T 1_{\{r < \tau_i\}} f(r, X_r^i, Y_r^i, Z_r^i, \mu_r^0, \nu_r^0) \, dr - 1_A_i \int_t^T Z_r \, dB_r.$$  

Adding them up and using the continuity of process $\mathcal{Y}$, we obtain that $P$-a.s.

$$Y_s = \eta_0 + \int_t^T 1_{\{r < \tau_i\}} f(r, X_r, Y_r, Z_r, \mu_r^0, \nu_r^0) \, dr - \int_t^T Z_r \, dB_r, \quad s \in [t, T].$$

Thus $\left( \mathcal{Y}, \mathcal{Z} \right) = \left( Y^{t, \xi_0, \mu^0, \nu^0}(\tau_0, \eta_0), Z^{t, \xi_0, \mu^0, \nu^0}(\tau_0, \eta_0) \right)$, proving (4.27).

Taking $\tau_i = T$ and $\eta_i = g(X_T^{t, \xi_i, \mu_i, \nu_i}) \in L^p(\mathcal{F}_T)$ for $i = 0, \cdots, n$, we see from (4.26) that

$$\sum_{i=1}^n 1_A_i \eta_i = \sum_{i=1}^n 1_A_i g(X_T^{t, \xi_i, \mu_i, \nu_i}) = \sum_{i=1}^n 1_A_i g(X_T^{t, \xi_0, \mu_0, \nu_0}) = \eta_0, \quad P\text{-a.s.}$$

Then (4.27) shows that $P$-a.s.

$$J(t, \xi_0, \mu^0, \nu^0) = Y_t^{t, \xi_0, \mu^0, \nu^0}(T, \eta_0) = \sum_{i=1}^n 1_A_i Y_t^{t, \xi_i, \mu_i, \nu_i} = \sum_{i=1}^n 1_A_i J(t, \xi_i, \mu_i, \nu_i). \quad \square$$

**Lemma 4.2.** Let $(t, x) \in [0, T] \times \mathbb{R}^k$ and $\varepsilon > 0$. For any $\beta \in \mathfrak{B}_t$, there exist $\{(A_n, \mu_n)\}_{n \in \mathbb{N}} \subset \mathcal{F}_t \times \mathcal{U}_t$ with $\lim_{n \to \infty} 1_{A_n} = 1$, $P$-a.s. such that for any $n \in \mathbb{N}$

$$J(t, x, \mu^n, \beta(\mu^n)) \geq (I(t, x, \beta) - \varepsilon) \land \varepsilon^{-1}, \quad P\text{-a.s. on } A_n,$$

where $I(t, x, \beta) \overset{\triangle}{=} \operatorname{esssup}_{\mu \in \mathcal{U}_t} J(t, x, \mu, \beta(\mu))$.

Similarly, there exist $\{(A_n, \beta_n)\}_{n \in \mathbb{N}} \subset \mathcal{F}_t \times \mathfrak{B}_t$ with $\lim_{n \to \infty} 1_{A_n} = 1$, $P$-a.s. such that for any $n \in \mathbb{N}$

$$w_1(t, x) \geq I(t, x, \beta_n) - \varepsilon, \quad P\text{-a.s. on } A_n.$$

**Proof:** (i) Let $\beta \in \mathfrak{B}_t$. Given $\mu^1, \mu^2 \in \mathcal{U}_t$, we set $A \overset{\triangle}{=} \{J(t, x, \mu^1, \beta(\mu^1)) \geq J(t, x, \mu^2, \beta(\mu^2))\} \subset \mathcal{F}_t$ and define $\tilde{\mu}_s \overset{\triangle}{=} 1_A \mu^1_s + 1_{A^c} \mu^2_s, s \in [t, T]$. Clearly, $\tilde{\mu}$ is an $\mathcal{F}$-progressively measurable process. For $i = 1, 2$, suppose that $E_T^{[T \mu^1_s]_U} ds < \infty$ for some $q_i > 2$. It follows that $E_T^{[T \mu^2_s]_U} ds \leq E_T^{[T \mu^1_s]_U} ds + E_T^{[T \mu^2_s]_U} ds < \infty$.

Thus, $\tilde{\mu} \in \mathcal{U}_t$. As $\tilde{\mu} = \mu^1$ on $[t, T] \times A$, taking $(\tau, A) = (t, A)$ in Definition 2.2 yields that $\beta(\tilde{\mu}) = \beta(\mu^1), ds \times dP$-a.s. on $[t, T] \times A$. Similarly, $\beta(\tilde{\mu}) = \beta(\mu^2), ds \times dP$-a.s. on $[t, T] \times A^c$. So $\beta(\tilde{\mu}) = 1_A \beta(\mu^1) + 1_{A^c} \beta(\mu^2), ds \times dP$-a.s. Then (4.28) shows that

$$J(t, x, \tilde{\mu}, \beta(\tilde{\mu})) = 1_A J(t, x, \mu^1, \beta(\mu^1)) + 1_{A^c} J(t, x, \mu^2, \beta(\mu^2)) = J(t, x, \mu^1, \beta(\mu^1)) \lor J(t, x, \mu^2, \beta(\mu^2)), \quad P\text{-a.s.},$$

which shows that the collection $\left\{J(t, x, \mu, \beta(\mu))\right\}_{\mu \in \mathcal{U}_t}$ is directed upwards. In light of Proposition VI-1-1 of [23], there exists a sequence $\left\{\tilde{\mu}^i\right\}_{i \in \mathbb{N}} \subset \mathcal{U}_t$ such that

$$I(t, x, \beta) = \operatorname{esssup}_{\mu \in \mathcal{U}_t} J(t, x, \mu, \beta(\mu)) = \lim_{i \to \infty} J(t, x, \tilde{\mu}^i, \beta(\tilde{\mu}^i)), \quad P\text{-a.s.} \quad (4.32)$$
So $I(t, x, \beta)$ is $\mathcal{F}_t$–measurable.

For any $j \in \mathbb{N}$, we set $\hat{A}_j \triangleq \{ J(t, x, \hat{\beta}(\mu^n)) \geq (I(t, x, \beta) - \varepsilon) \wedge \varepsilon^{-1} \} \in \mathcal{F}_t$ and $\hat{A} \triangleq \bigcup_{j < i} \hat{A}_j \in \mathcal{F}_t$. Fix $n \in \mathbb{N}$ and set $A_n \triangleq \bigcup_{i=1}^{n} \hat{A}_i \in \mathcal{F}_t$. Similar to $\mu^n \triangleq \sum_{i=1}^{n} \hat{A}_i \mu^n + 1_{A_n} \mu^n$ also defines a $\mathcal{U}_t$–process. For $i = 1, \cdots, n$, as $\mu^n = \mu^1$ on $[t, T] \times \hat{A}_i$, taking $(\tau, A) = (t, \hat{A}_i)$ in Definition 2.2 shows that $\beta(\mu^n) = \beta(\mu^1)$. Then (4.28) implies that $1_{A_i} J(t, x, \mu^n, \beta(\mu^n)) = 1_{A_i} J(t, x, \mu^1, \beta(\mu^1))$, $P$–a.s. Adding them up over $i \in \{1, \cdots, n\}$ gives

$$1_{A_n} J(t, x, \mu^n, \beta(\mu^n)) = \sum_{i=1}^{n} 1_{A_i} J(t, x, \mu^1, \beta(\mu^1)) \geq 1_{A_n} ((I(t, x, \beta) - \varepsilon) \wedge \varepsilon^{-1}), \quad P$–a.s.$$

Let $\mathcal{N}$ be the $P$–null set such that (4.32) holds on $\mathcal{N}^c$. Clearly, $\{ I(t, x, \beta) < \infty \} \cap \mathcal{N}^c \subset \bigcup_{i \in \mathbb{N}} \{ J(t, x, \mu^1, \beta(\mu^1)) \geq I(t, x, \beta) - \varepsilon \}$ and $\{ I(t, x, \beta) = \infty \} \cap \mathcal{N}^c \subset \bigcup_{i \in \mathbb{N}} \{ J(t, x, \mu^1, \beta(\mu^1)) \geq \varepsilon^{-1} \}$. It follows that

$$\mathcal{N}^c \subset \bigcup_{i \in \mathbb{N}} \{ J(t, x, \mu^1, \beta(\mu^1)) \geq I(t, x, \beta) - \varepsilon \} \cup \{ J(t, x, \mu^1, \beta(\mu^1)) \geq \varepsilon^{-1} \} = \bigcup_{i \in \mathbb{N}} \hat{A}_i = \bigcup_{i \in \mathbb{N}} \hat{A}_i = \bigcup_{i \in \mathbb{N}} A_n.$$

So $\lim_{n \to \infty} \uparrow 1_{A_n} = 1$, $P$–a.s.

(ii) Let $\beta_1, \beta_2 \in \mathcal{B}_t$. We just showed that $I(t, x, \beta_1)$ and $I(t, x, \beta_2)$ are $\mathcal{F}_t$–measurable, so $\mathcal{A}_n \triangleq \{ I(t, x, \beta_1) \leq I(t, x, \beta_2) \}$ belongs to $\mathcal{F}_t$. For any $\mu \in \mathcal{U}_t$, similar to $\hat{\mu}$ above, $\beta_0(\mu) \hat{\triangleq} 1_{A_n} \beta_1(\mu) + 1_{A_n} \beta_2(\mu)$ defines a $\mathcal{V}_t$–process. For $i = 1, 2$, letting $C_i > 0$ be the constant associated to $\beta_i$ in Definition 2.2 (i), we see that

$$[(\beta_0(\mu))_s]_\mathcal{V} = 1_{A_n} [(\beta_1(\mu))_s]_\mathcal{V} + 1_{A_n} [(\beta_2(\mu))_s]_\mathcal{V} \leq \kappa + (C_1 \lor C_2) [\mu]_\mathcal{U}, \quad ds \times dP - a.s.$$

Let $\mu^1, \mu^2 \in \mathcal{U}_t$ such that $\mu^1 = \mu^2$, $ds \times dP$–a.s. on $[t, \tau] \cup [\tau, T]$. Fix $\mathcal{A}_n \triangleq \{ I(t, x, \beta_1) \leq I(t, x, \beta_2) \} \in \mathcal{F}_t$. By Definition 2.2 (ii) $\beta_1(\mu^1) = \beta_1(\mu)$ and $\beta_2(\mu^1) = \beta_2(\mu^2)$, $ds \times dP$–a.s. on $[t, \tau] \cup [\tau, T].$ Then it follows that for $ds \times dP$–a.s. $(s, \omega) \in [t, \tau] \cup [\tau, T]$, $A_n$.

$$(\beta_0(\mu^1))_s(\omega) = 1_{A_n} (\beta_1(\mu^1))_s(\omega) + 1_{A_n} (\beta_2(\mu^1))_s(\omega) = 1_{A_n} (\beta_1(\mu^2))_s(\omega) + 1_{A_n} (\beta_2(\mu^2))_s(\omega) = (\beta_0(\mu^2))_s(\omega). \quad (4.33)$$

Hence, $\beta_0 \in \mathcal{B}_t$.

For any $\mu \in \mathcal{U}_t$, (4.28) shows that $J(t, x, \mu, \beta_0(\mu)) = 1_{A_n} J(t, x, \mu, \beta_1(\mu)) + 1_{A_n} J(t, x, \mu, \beta_2(\mu))$, $P$–a.s. Then taking essential supremum over $\mu \in \mathcal{U}_t$ and using Lemma 2.4 (2) yield that

$$I(t, x, \beta_0) = 1_{A_n} I(t, x, \beta_1) + 1_{A_n} I(t, x, \beta_2) = I(t, x, \beta_1) \wedge I(t, x, \beta_2), \quad P$–a.s.$$

Thus the collection $\{ I(t, x, \beta) \}_\beta \in \mathcal{B}_t$, is directed downwards. By Proposition VI-1-1 of [23] again, one can find a sequence $\{ \beta_i \}_i \in \mathcal{B}_t$ such that

$$w_i(t, x) = \text{ess inf}_t \{ I(t, x, \beta) \} = \lim_{i \to \infty} I(t, x, \beta_i), \quad P$–a.s.$$

For any $i \in \mathbb{N}$, we set $\hat{A}_i \triangleq \{ I(t, x, \beta_i) \leq w_i(t, x) + \varepsilon \} \in \mathcal{F}_t$ and $\hat{A} \triangleq \bigcup_{j < i} \hat{A}_j \in \mathcal{F}_t$. Fix $n \in \mathbb{N}$ and set $\mathcal{A}_n \triangleq \bigcup_{i=1}^{n} \hat{A}_i \in \mathcal{F}_t$. For any $\mu \in \mathcal{U}_t$, similar to $\hat{\mu}$ above, $\beta_0(\mu) \hat{\triangleq} \sum_{i=1}^{n} 1_{A_i} \beta_i(\mu) + 1_{A_n} \beta_i(\mu)$ defines a $\mathcal{V}_t$–process. For $i = 1, \cdots, n$, let $C_i > 0$ be the constant associated to $\beta_i$ in Definition 2.2 (i). Setting $C_n \hat{\triangleq} \max\{C_i : i = 1, \cdots, n\}$, we can deduce that

$$[(\beta_0(\mu))_s]_\mathcal{V} = \sum_{i=1}^{n} 1_{A_i} [(\beta_i(\mu))_s]_\mathcal{V} + 1_{A_n} [(\beta_i(\mu))_s]_\mathcal{V} \leq \kappa + C_n [\mu]_\mathcal{U}, \quad ds \times dP - a.s.$$

Let $\mu^1, \mu^2 \in \mathcal{U}_t$ such that $\mu^1 = \mu^2$, $ds \times dP$–a.s. on $[t, \tau] \cup [\tau, T]$. Then it holds for $ds \times dP$–a.s. $(s, \omega) \in [t, \tau] \cup [\tau, T]$ that

$$(\beta_n(\mu^1))_s(\omega) = \sum_{i=1}^{n} 1_{A_i} (\beta_i(\mu^1))_s(\omega) + 1_{A_n} (\beta_i(\mu^1))_s(\omega) = \sum_{i=1}^{n} 1_{A_i} (\beta_i(\mu^2))_s(\omega) + 1_{A_n} (\beta_i(\mu^2))_s(\omega) = (\beta_n(\mu^2))_s(\omega).$$
So $\beta_n \in \mathcal{B}_1$. For any $\mu \in \mathcal{U}_t$, applying (4.28) again yields $\mathbf{1}_{A_n} J(t, x, \mu, \beta_n(\mu)) = \sum_{i=1}^{n} \mathbf{1}_{\tilde{\mathcal{A}}_i} J(t, x, \tilde{\beta}_i(\mu))$, $P$–a.s. Taking essential supremum over $\mu \in \mathcal{U}_t$ and using Lemma 2.4 (2) again yield that

$$\mathbf{1}_{A_n} I(t, x, \beta_n) = \sum_{i=1}^{n} \mathbf{1}_{\tilde{\mathcal{A}}_i} I(t, x, \tilde{\beta}_i) \leq \mathbf{1}_{A_n} (w(t, x) + \varepsilon), \quad P$–a.s.$$

Let $\tilde{N}$ be the $P$–null set such that (4.34) holds on $\tilde{N}^c$. As $|w(t, x)| < \infty$ by Proposition 2.1 and Proposition 2.2 we see that $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{i \in \mathbb{N}} \tilde{\mathcal{A}}_i = \bigcup_{i \in \mathbb{N}} \tilde{\mathcal{A}}_i = \tilde{N}$.

**Proof of Theorem 2.1** 1) For any $m \in \mathbb{N}$ and $(s, \tau) \in [t, T] \times \mathbb{R}^k$, the continuity of $\phi$, $\tilde{\phi}$ shows that there exists a $\delta_{m, n}^0 \in (0, 1/m)$ such that

$$|\phi(s', \tau') - \phi(s, \tau)| + |\tilde{\phi}(s', \tau') - \tilde{\phi}(s, \tau)| \leq \frac{1}{m}, \quad \forall (s', \tau') \in [(s - \delta_{m, n}^0) \vee t, (s + \delta_{m, n}^0) \wedge T] \times \mathcal{O}_{\eta_{m, j}}(\tau).$$

(4.35)

By classical covering theory, $\{\mathcal{D}_m(s, \tau) \triangleq (s - \delta_{m, n}^0, s + \delta_{m, n}^0) \times O_{\eta_{m, j}}(\tau)\}_{s, \tau} \in [t, T] \times \mathbb{R}^k$ has a finite subcollection $\{\mathcal{D}_m(s_i, \tau_i)\}_{i=1}^{N_m}$ to cover $\mathcal{O}_m(t, x)$. For $i = 1, \ldots, N_m$, we set $t_i \triangleq (s_i + \delta_{m, n}^0, s_i, \tau_i) \in T$.

1a) Fix $(\beta, \mu) \in \mathcal{B}_3 \times \mathcal{U}_t$ and simply denote $\beta_{t, m} \mu$ by $\beta$. By Lemma 2.1, $\tilde{\mu}_n \triangleq \mathbf{1}_{(s \leq t \leq r)} \mathbf{1}_{(\pi \geq r)} \mu_s + \mathbf{1}_{(s \pi \geq T)} u_0, s \in [t, T]$ defines a $\mathcal{U}_t$–control. We set $\Theta \triangleq (t, x, \mu, \beta(\mu))$ and $\Theta \triangleq (t, x, \tilde{\mu}, \tilde{\beta}(\tilde{\mu}))$.

For any $s \in [t, T]$ and $\tilde{\mu}$, the process $(\mu, \tilde{\mu})$, $\mathbf{1}_{(r \leq s \leq r)} \mathbf{1}_{(r \geq s \leq T)} \mu_s + \mathbf{1}_{(r \geq s \leq T)} \tilde{\mu}_s$, $r \in [t, T]$ is clearly $\mathbf{F}$–progressively measurable. Suppose that $E \int_t^T [\mu_s]^{q} ds + E \int_t^T [\tilde{\mu}_s]^{\tilde{q}} ds < \infty$ for some $q > 2$ and $\tilde{q} > 2$. It follows that

$$E \int_t^T \left( [\tilde{\mu} + \tilde{\mu}]_r^{\tilde{q}} \right) dr \leq E \int_t^T \left( [\mathbf{1}_{[\tilde{\mu} + \tilde{\mu}]_r}]^{\tilde{q}} \right) dr + E \int_t^T [\tilde{\mu}]^{\tilde{q}} \mu_r dr < \infty.$$

Thus, $\tilde{\mu} + \tilde{\mu} \in \mathcal{U}_t$. Then we can define

$$\beta^*(\tilde{\mu}) \triangleq [\beta(\tilde{\mu} + \tilde{\mu})]_r \in \mathcal{V}_s.$$

(4.36)

For $dr \times dP$–a.s. $(r, \omega) \in [s, T] \times \Omega$,

$$[(\beta^*(\tilde{\mu}))_r]_\omega = \left[ (\beta(\tilde{\mu} + \tilde{\mu}))_r \right]_\omega \leq \kappa + C_{s, \delta} \left[ [\beta(\tilde{\mu} + \tilde{\mu})]_r \right]_\omega = \kappa + C_{s, \delta} [\beta]_r \omega.$$

Let $\tilde{\mu}^1, \tilde{\mu}^2 \in \mathcal{U}_t$ such that $\tilde{\mu}^1 = \tilde{\mu}^2$, $dr \times dP$–a.s. on $[S, \mathcal{F}]_A$ for some $s \in S, \tau, \mathcal{A} \in \mathcal{F}_t$. Then $\tilde{\mu} + \tilde{\mu}^1 = \tilde{\mu} + \tilde{\mu}^2$, $dr \times dP$–a.s. on $[t, \mathcal{F}]_A$. By Definition 2.2, $\beta(\tilde{\mu} + \tilde{\mu}^1) = \beta(\tilde{\mu} + \tilde{\mu}^2), dr \times dP$–a.s. on $[t, \mathcal{F}]_A$. It follows that for $dr \times dP$–a.s. $(r, \omega) \in [s, \mathcal{F}]_A$

$$\left( \beta^*(\tilde{\mu}^1) \right)_r (\omega) = \left( \beta(\tilde{\mu} + \tilde{\mu}^1) \right)_r (\omega) = \left( \beta(\tilde{\mu} + \tilde{\mu}^2) \right)_r (\omega) = \left( \beta^*(\tilde{\mu}^2) \right)_r (\omega).$$

Hence, $\beta^* \in \mathcal{B}_s$.

Fix $m \in \mathbb{N}$ with $m \geq C_{x, \delta} \triangleq \sup \{ |\phi(s, \tau)| : (s, \tau) \in \mathcal{O}_{\delta, \gamma}(t, x) \cap ([t, T] \times \mathbb{R}^k) \}$. Given $i = 1, \ldots, N_m$, (4.30) shows that there exists $\left( A_{n, m}^i, \tilde{\mu}_{m, n}^i \right) \in \mathcal{C}_F^i$ such that $\mathbf{1}_{A_{n, m}^i} = 1, P$–a.s. such that for any $n \in \mathbb{N}$

$$J(t_i, x_i, \mu_{n, m}^i, \beta^*(\mu_{n, m}^i)) \geq (I(t_i, x_i, \beta^*(\mu_{n, m}^i)) - 1/m) \wedge m, \quad P$–a.s. on $A_{n, m}^i.$$

(4.37)

As $Y_{\bar{\Theta}}(T, g(X_{\bar{\Theta}})) \in \mathcal{C}_F([t, T])$, the Monotone Convergence Theorem shows that

$$\lim_{n \to \infty} E \left[ \left( \sup_{s \in [t, T]} Y_{\bar{\Theta}}(T, g(X_{\bar{\Theta}})) \right)^p + (C_{s, \delta})^p \right] = 0.$$

So there exists an $n(m, i) \in \mathcal{N}$ such that $E \left[ \left( \sup_{s \in [t, T]} Y_{\bar{\Theta}}(T, g(X_{\bar{\Theta}})) \right)^p + (C_{s, \delta})^p \right] \leq m^{-(1+p)} N_{n, m}^{-1}$. Set $A_{n, m}^i = A_{n, m}^i (\mu_{n, m}^i, \tilde{\mu}_{m, n}^i)$ and $\tilde{A}_m^i = \{ (\tau, X_{\bar{\Theta}}^\tau) : \tilde{\mathcal{D}}_m(s_i, x_i) / \tilde{\mathcal{D}}_m(s_j, x_j) \} \in \mathcal{F}_t$. As $\tilde{A}_m^i \subset \{ (\tau, X_{\bar{\Theta}}^\tau) \in \mathcal{D}_m(s_i, x_i) \}$ we see that $\tilde{A}_m^i \subset \{ (\tau, X_{\bar{\Theta}}^\tau) \in \mathcal{D}_m(s_i, x_i) \}$
By the continuity of process $X^\Theta$, $(\tau, X^\Theta) \in \partial\mathcal{O}_\delta(t, x)$, $P$–a.s. So $\{\tilde{A}_m\}_{m=1}^N$ forms a partition of $\mathcal{N}$ for some $P$–null set $\mathcal{N}$. Then we can define an $F$–stopping time $\tau_m \triangleq \sum_{i=1}^{N_m} 1_{[t, T]}^i + 1_N T \geq \tau$ as well as a process

$$
\mu^m_s \triangleq 1_{(s < \tau_m)} \tilde{\mu}_s + 1_{(s \geq \tau_m)} \left( \sum_{i=1}^{N_m} 1_{[t, T]}^i (\mu^m_i)_s + 1_{A_m} \tilde{\mu}_s \right)
$$

$$
= 1_{A_m} \tilde{\mu}_s + \sum_{i=1}^{N_m} 1_{\tilde{A}_m \cap [t, T]} (1_{(s < t_i)} \tilde{\mu}_s + 1_{(s \geq t_i)} (\mu^m_i)_s), \quad \forall s \in [t, T],
$$

where $A_m \triangleq \left( \bigcup_{i=1}^{N_m} (\tilde{A}_i \cap [t, T]) \right) \cup \mathcal{N}$.

Let $s \in [t, T]$ and $U \in \mathcal{B}(\mathbb{R})$. As $[t, \tau] \in \mathcal{F}$, we see that $\mathcal{D} \supseteq [t, \tau] \cap ([t, s] \times \Omega) \in \mathcal{B}([t, s]) \otimes \mathcal{F}_s$. The $F$–progressive measurability of $\tilde{\mu}$ then implies that

$$
\{(r, \omega) \in \mathcal{D}: \mu^m_r(\omega) \in U \} = \{(r, \omega) \in \mathcal{D}: \tilde{\mu}_r(\omega) \in U \} = \mathcal{D} \cap \{(r, \omega) \in [t, s] \times \Omega: \tilde{\mu}_r(\omega) \in U \} \in \mathcal{B}([t, s]) \otimes \mathcal{F}_s.
$$

Given $i = 1, \ldots, N_m$, we set $\tilde{A}_m \triangleq (\tilde{A}_i \cap [t, T]) \cup \mathcal{N} \in \mathcal{F}_t$. If $s < t_i$, both $\mathcal{D}_m^{\cap} \triangleq \cap [t, s] \times (\tilde{A}_i \cap [t, T])$ and $\mathcal{D}_m \triangleq \cap [t, s] \times (\tilde{A}_i \cap [t, T])$ are empty. Otherwise, if $s \geq t_i$, both $\mathcal{D}_m = [t, s] \times (\tilde{A}_i \cap [t, T])$ and $\mathcal{D}_m^{\cap} = [t, s] \times \mathcal{N}$ belong to $\mathcal{B}([t, s]) \otimes \mathcal{F}_s$. Using a similar argument to (4.38) on the $F$–progressive measurability of process $\mu^m$ yields that

$$
\{(r, \omega) \in \mathcal{D}_i^m: \mu^m_r(\omega) \in U \} = \{(r, \omega) \in \mathcal{D}_i: (\mu^m_i)_r(\omega) \in U \} \in \mathcal{B}([t, s]) \otimes \mathcal{F}_s.
$$

and

$$
\{(r, \omega) \in \mathcal{D}_i: \mu^m_r(\omega) \in U \} = \{(r, \omega) \in \mathcal{D}: \tilde{\mu}_r(\omega) \in U \} \in \mathcal{B}([t, s]) \otimes \mathcal{F}_s,
$$

both of which together with (4.38) shows the $F$–progressive measurability of $\mu^m$. For $i = 1, \ldots, N_m$, suppose that $E \int_{T}^{T} [(\mu^m_i)_r^q]_U^q d\tau < \infty$ for some $q_i > 2$. Setting $q_s \equiv q \land \min\{q_i: i = 1, \ldots, N_m\}$, we can deduce that

$$
E \int_t^T [\mu^m_r]^q U d\tau \leq E \int_t^T [\mu^m_r]^q U d\tau + \sum_{i=1}^{N_m} E \int_{T}^{T} [(\mu^m_i)_r^q U]_U^q d\tau < \infty.
$$

Hence, $\mu^m \in \mathcal{U}_t$.

Next, we set $\Theta_m \triangleq (t, x^\mu, \beta(\mu^m))$. As $\mu^m = \mu = (t, A)$ in Definition 2.2 shows that $\beta(\mu^m) = \beta(\mu)$, $\mu^m \in \mathcal{P}$–a.s. on $[t, \tau]$, and then applying (2.7) with $(\tau, A) = (t, \emptyset)$ yields that $P$–a.s.

$$
X^\Theta_t = X^\Theta_s \in \mathcal{U}_\delta(x), \quad \forall s \in [t, \tau].
$$

Thus, for any $\eta \in \mathbb{L}^p(\mathcal{F}_\tau)$, the BSDE ($t, \eta, f^\Theta_t$) and the BSDE ($t, \eta, f^\Theta_t$) are essentially the same. To wit,

$$
(Y^\Theta_t(\tau, \eta), Z^\Theta_t(\tau, \eta)) = (Y^\Theta_t(\tau, \eta), Z^\Theta_t(\tau, \eta)).
$$

Given $A \in \mathcal{F}_t$, we see from (4.39) that

$$
1_A X^\Theta_{\tau \land s} = 1_A X^\Theta_{\tau \land s} + 1_A \int_{\tau \land s}^\tau b(r, X^\Theta_{\tau \land r}, \mu^m_r, (\beta(\mu^m)_r))dr + 1_A \int_{\tau \land s}^\tau \sigma(r, X^\Theta_{\tau \land r}, \mu^m_r, (\beta(\mu^m)_r))dB_r,
$$

$$
= 1_A X^\Theta_{\tau \land s} + 1_A \int_{\tau \land s}^\tau b(r, X^\Theta_{\tau \land r}, \mu^m_r, (\beta(\mu^m)_r))dr + 1_A \int_{\tau \land s}^\tau \sigma(r, X^\Theta_{\tau \land r}, \mu^m_r, (\beta(\mu^m)_r))dB_r, \quad s \in [t, T].
$$

It follows that

$$
1_A \sup_{r \in [t, s]} \left| X^\Theta_{\tau \land r} - X^\Theta_{\tau \land r} \right| \leq \int_{\tau \land s}^\tau \left| b(r, X^\Theta_{\tau \land r}, \mu^m_r, (\beta(\mu^m)_r))dr \right| + \sup_{r \in [t, s]} \left| \int_{\tau \land s}^\tau \sigma(r, X^\Theta_{\tau \land r}, \mu^m_r, (\beta(\mu^m)_r))dB_r \right|, \quad s \in [t, T].
$$
Let $C(\kappa, x, \delta)$ denote a generic constant, depending on $\kappa + |x| + \delta$, $C^\circ_{x, \delta}$, $T$, $\gamma$, $p$ and $|g(0)|$, whose form may vary from line to line. Squaring both sides of (4.41) and taking expectation, we can deduce from Hölder’s inequality, Doob’s martingale inequality, (2.1), (2.2), (4.39) and Fubini’s Theorem that

\[
E\left[ \mathbf{1}_A \sup_{r \in [t, s]} |X^\Theta_{m \wedge r} - X^\Theta_{\tau \wedge r}|^2 \right] \\
\leq 4E \int_{\tau \wedge s}^{\tau_m \wedge s} |1_A b(r, X^\Theta_{m \wedge r}, u_0, (\beta(\mu^m)))|^2 dr + 8E \int_{\tau \wedge s}^{\tau_m \wedge s} |1_A \sigma(r, X^\Theta_{m \wedge r}, u_0, (\beta(\mu^m)))|^2 dr \\
\leq 12\gamma^2 E \int_{\tau \wedge s}^{\tau_m \wedge s} 1_A \left( |X^\Theta_{m \wedge r} - X^\Theta_{\tau \wedge r}| + |X^\Theta_{\tau \wedge r}| + 1 + \left[ (\beta(\mu^m))_{r \wedge s} \right]^2 \right) dr \\
\leq 24\gamma^2 \int_s^t E \left[ \mathbf{1}_A \sup_{r \in [t, r]} \left| X^\Theta_{m \wedge r} - X^\Theta_{\tau \wedge r} \right|^2 \right] dr + \frac{C(\kappa, x, \delta)}{m} P(A), \quad \forall s \in [t, T],
\]

where we used the facts that

\[
\tau_m - \tau \leq \sum_{i=1}^{N_m} 2\delta^m_{s,t_i,x_i} < \frac{2}{m}, \quad P\text{-a.s. and } [(\beta(\mu^m))_{r \wedge s}] \leq \kappa, \quad dr \times dP - \text{a.s. on } \tau, \tau_m. \tag{4.43}
\]

Then an application of Gronwall’s inequality yields that

\[
E\left[ \mathbf{1}_A \sup_{r \in [t, s]} |X^\Theta_{m \wedge r} - X^\Theta_{\tau \wedge r}|^2 \right] \leq \frac{C(\kappa, x, \delta)}{m} P(A)e^{2\gamma^2(s-t)}, \quad \forall s \in [t, T].
\]

In particular, \( E\left[ \mathbf{1}_A \sup_{r \in [t, T]} \left| X^\Theta_{m \wedge r} - X^\Theta_{\tau \wedge r} \right|^2 \right] \leq \frac{C(\kappa, x, \delta)}{m} P(A). \) Letting $A$ vary in $\mathcal{F}_t$ yields that

\[
E\left[ \sup_{r \in [t, T]} \left| X^\Theta_{m \wedge r} - X^\Theta_{\tau \wedge r} \right|^2 |\mathcal{F}_t \right] \leq \frac{C(\kappa, x, \delta)}{m}, \quad P\text{-a.s.} \tag{4.44}
\]

Let $i = 1, \cdots, N_m$ and set $\Theta^i_m \triangleq \left[ (t_i, X^\Theta_{t_i}, [\mu^m]_{t_i})^t \right]$ and $[\beta(\mu^m)]_t = \left[ (\beta(\mu^m))_{r \wedge s} \right]$. We see from (2.6) that $X^\Theta_{t_i} = X^\Theta_{t_i}^i$, $P$-a.s. It then follows from (2.12) that

\[
Y^\Theta_{t_i}(T, g(X^\Theta_{t_i})) = Y^\Theta_{t_i}(T, g(X^\Theta_{t_i})) = Y^\Theta_{t_i}(T, g(X^\Theta_{t_i}^i)) = J(\Theta^i_m), \quad P\text{-a.s.} \tag{4.45}
\]

Similar to $\mu^m$,

\[
(\tilde{\mu}^m)^s = 1_{s < \tau_m} \tilde{\mu}_s + 1_{s \geq \tau_m} \left(1_{ \tilde{A}^m \cap \lambda^m}(\mu^m)_s + 1_{(A^m \cap \lambda^m)^c}(\mu^m)_s \right) \quad s \in [t, T]
\]

also defines a $U_t$-process. As $\mu^m = \tilde{\mu}^m$ on $[t, \tau_m \cup [\tau_m, T] \cap \lambda^m \cap A^m$ and $\tilde{\mu}^m = \tilde{\mu}^m \cup \tilde{\mu}^m$ on $([t, \tau)_t \times \Omega) \cup ([t, T] \times (\tilde{A}^m \cap \lambda^m \cap A^m))$, Definition 2.2 shows that $\beta(\mu^m) = \beta(\tilde{\mu}^m)$, $ds \times dP$-a.s. on $[t, \tau_m \cup [\tau_m, T] \cap \lambda^m \cap A^m$ and $\beta(\tilde{\mu}^m) = \beta(\tilde{\mu}^m \cup \tilde{\mu}^m)$, $ds \times dP$-a.s. on $([t, \tau)_t \times \Omega) \cup ([t, T] \times (\tilde{A}^m \cap \lambda^m \cap A^m)$). Thus $\mu^m, \beta(\mu^m) = (\tilde{\mu}^m \cup \tilde{\mu}^m, \beta(\tilde{\mu}^m \cup \tilde{\mu}^m)))$, $ds \times dP$-a.s. on $[t, \tau_m \cup [\tau_m, T] \cap \lambda^m \cap A^m$. From (4.36), one has $([\mu^m]_{t_i}, [\beta(\mu^m)]_{t_i}) = (\mu^m_{t_i}, \beta_{t_i}(\mu^m_{t_i}))$, $ds \times dP$-a.s. on $[t_i, T] \times (\tilde{A}^m \cap \lambda^m \cap A^m)$. Then by (4.45), (4.28) and (2.13), it holds $P$-a.s. on $\tilde{A}^m \cap \lambda^m \subset \mathcal{F}_t$ that

\[
Y^\Theta_{t_i}(T, g(X^\Theta_{t_i}^i)) = J(t_i, X^\Theta_{t_i}, [\mu^m]_{t_i}, [\beta(\mu^m)]_{t_i}) \geq J(t_i, X^\Theta_{t_i}, [\mu^m_{t_i}, [\beta(\mu^m)]_{t_i}) - c_0|X^\Theta_{t_i} - X^\Theta_{t_i}^i|^p.
\]

Since $\mathcal{D}_m(s_i, x_i) \subset \sigma(\tilde{A}^m(t, x))$, it is easy to see that

\[
\mathcal{D}_m(s_i, x_i) = [s_i - \delta^m_{s_i, x_i}, s_i + \delta^m_{s_i, x_i}] \subset \sigma(\tilde{A}^m_{s_i, x_i}(t, x)) \subset \sigma(\tilde{A}^m + 2\sqrt{3}\delta^m_{s_i, x_i}(t, x)) \subset \sigma(\tilde{A}^m + 2\sqrt{3}(t, x)), \quad \mathcal{D}_m(s_i, x_i)
\]

So $\phi(t_i, x_i) \leq C^\circ_{x, \delta} < m + 1/m$. On the other hand, one has $\phi(t_i, x_i) \leq w_1(t_i, x_i) \leq I(t_i, x_i, \beta_{t_i})$, $P$-a.s. Then it follows from (4.37) that

\[
\phi(t_i, x_i) \leq I(t_i, x_i, \beta_{t_i}) \land (m + 1/m) \leq J(t_i, x_i, [\mu^m_{t_i}, [\beta(\mu^m)]_{t_i}) + 1/m, \quad P\text{-a.s. on } A^m.
\]
As $|X^n_{i\tau} - X^i_\tau|^{2/p} < (\delta^{m,n}_{s,x})^{2/p} < m^{-2/p} \leq 1/m$ on $\tilde{A}^m_{i\tau}$, we can also deduce from (2.13), (4.35) and the continuity of $\phi$ that it holds $P$-a.s. on $\tilde{A}^m_{i\tau} \cap A^m_{i\tau}$ that

$$J(t_i, X^\Phi_{r_i}, \mu^m_{r_i}, \beta^m_{r_i}(\mu^m_{r_i})) \geq J(t_i, x_i, \mu^m_{r_i}, \beta^m_{r_i}(\mu^m_{r_i})) - \frac{c_0}{m} \geq \phi(t_i, x_i) - \frac{c_0}{m} \geq \phi(x_i, x_i) - \frac{c_0}{m} \geq \phi(r, X^\Phi_{r_i}) - \frac{c_0}{m} \triangleq \eta_m \in \mathbb{L}^{\infty}(F_{\tau}).$$

Thus it holds $P$-a.s. on $\bigcup_{i=1}^N (\tilde{A}^m_{i\tau} \cap A^m_{i\tau})$ that

$$Y^\Theta_{r_m}(T, g(X^\Theta_{T^m})) \geq \eta_m - c_0|X^\Theta_{r_m} - X^\Phi_{r_i}|^{2/p} \triangleq \eta_m \in \mathbb{L}^p(F_{\tau_m}).$$

(4.46)

By (2.9), it holds $P$-a.s. that

$$|Y^\Theta_{r}(\tau, \eta_m) - Y^\Theta_{r}(\tau, \phi(\tau, X^\Theta_{r}))|^p \leq c_0 E \left[ |\eta_m - \phi(\tau, X^\Theta_{r})|^p \right] \leq \frac{c_0}{m^p}. \quad (4.47)$$

Let $(Y^m, Z^m) \in \mathbb{G}^p_F([t, T])$ be the unique solution of the following BSDE with zero generator:

$$Y^m_s = Y^\Theta_m(\tau_m, \eta_m) - \int_s^T Z^m_r dB_r, \quad s \in [t, T].$$

For any $s \in [t, T]$, one can deduce that

$$Y^m_{\tau \wedge s} = E[Y^m_{\tau \wedge s}|F_{\tau}] = E \left[ Y^\Theta_m(\tau_m, \eta_m) - \int_{\tau \wedge s}^T Z^m_r dB_r | F_{\tau} \right] = Y^\Theta_m(\tau_m, \eta_m) - \int_{\tau \wedge s}^T Z^m_r dB_r, \quad P$-a.s.$$

By the continuity of process $Y^m$, it holds $P$-a.s. that

$$Y^m_{\tau \wedge s} = Y^\Theta_m(\tau_m, \eta_m) - \int_{\tau \wedge s}^T Z^m_r dB_r = Y^\Theta_m(\tau_m, \eta_m) - \int_{s}^{\tau} 1_{\{r < \tau\}} Z^m_r dB_r, \quad s \in [t, T]. \quad (4.48)$$

Thus, we see that $(Y^m_{\tau \wedge s}, Z^m_{\tau \wedge s}) = (Y^m_{\tau \wedge s}, 1_{\{s < \tau\}} Z^m_{\tau \wedge s}), s \in [t, T]$. Also, taking $[\cdot|F_{\tau \wedge s}]$ in (4.48) shows that $P$-a.s.

$$Y^m_s = Y^\Theta_m(\tau_m, \eta_m) = E \left[ Y^\Theta_m(\tau_m, \eta_m) | F_{\tau \wedge s} \right], \quad \forall s \in [t, T].$$

On the other hand, let $(\tilde{Y}^m, \tilde{Z}^m) \in \mathbb{G}^p_F([t, T])$ be the unique solution of the following BSDE with zero generator:

$$\tilde{Y}^m_s = \eta_m - \int_s^T \tilde{Z}^m_r dB_r, \quad s \in [t, T].$$

(4.49)

Similar to $(Y^m, Z^m)$, it holds $P$-a.s. that

$$(\tilde{Y}^m_s, \tilde{Z}^m_s) = (Y^m_{\tau \wedge s}, 1_{\{s < \tau\}} Z^m_{\tau \wedge s}) \quad \text{and} \quad \tilde{Y}^m_s = E[\eta_m|F_{\tau \wedge s}], \quad \forall s \in [t, T]. \quad (4.50)$$

We can deduce that $(Y^m, Z^m) \overset{\Delta}{=} \left\{ (1_{\{s < \tau\}} Y^m_s + 1_{\{s \geq \tau\}} Y^\Theta_m(\tau_m, \eta_m), 1_{\{s < \tau\}} Z^m_s + 1_{\{s \geq \tau\}} Z^\Theta_m(\tau_m, \eta_m)) \right\}_{s \in [t, T]} \in \mathbb{G}^p_F([t, T])$ solves the following BSDE

$$Y^m_s = 1_{\{s \geq \tau\}} Y^\Theta_m(\tau_m, \eta_m) + 1_{\{s < \tau\}} Y^\Theta_m(\tau_m, \eta_m) - \int_{\tau \wedge s}^T Z^m_r dB_r = Y^\Theta_m(\tau_m, \eta_m) - \int_{\tau \wedge s}^T 1_{\{r < \tau\}} Z^m_r dB_r$$

$$= \eta_m + \int_{\tau \wedge s}^T f^\Theta_{r_m}(r, Y^\Theta_m(\tau_m, \eta_m), Z^\Theta_m(\tau_m, \eta_m)) dr - \int_{\tau \wedge s}^T Z^\Theta_m(\tau_m, \eta_m) dB_r = \int_{\tau \wedge s}^T 1_{\{r \geq \tau\}} Z^m_r dB_r, \quad s \in [t, T]. \quad (4.51)$$

Since (2.4), Hölder’s inequality and (2.8) imply that

$$E \left[ \int_t^T 1_{\{s \geq \tau\}} \left| f^\Theta_m(s, \tilde{Y}^m_s, \tilde{Z}^m_s) \right|^p ds \right] \leq c_p E \left[ \int_t^T \left| f^\Theta_m(s, 0, 0) \right|^p ds + \sup_{s \in [t, T]} |\tilde{Y}^m_s|^p + \left( \int_t^T |\tilde{Z}^m_s|^2 ds \right)^{p/2} \right] < \infty.$$
applying (1.5) to $\mathcal{Y}^m - \tilde{\mathcal{Y}}^m$ and using (4.50) yield that

\[
E\left[|Y_t^{\Theta^m}(\tau_m, \eta_m) - \eta_m|^p \mid \mathcal{F}_t \right] = E\left[|Y_t^{\Theta^m} - \tilde{Y}_t^{\Theta^m}|^p \mid \mathcal{F}_t \right] \leq E\left[ \sup_{s \in [t,T]} |\mathcal{Y}_s^{m} - \tilde{\mathcal{Y}}_s^{m}|^p \mid \mathcal{F}_t \right] \leq c_0 E\left[ \int_{\tau}^T |f_{\tau_m}^{\Theta^m}(s, \tilde{Y}_s^{m}, \tilde{Z}_s^{m})|^p ds \mid \mathcal{F}_t \right]
\]

\[
= c_0 E\left[ \int_{\tau}^T \left| f(s, X_{\tau_m}^{\Theta^m}, \eta_m, 0, u_0, (\beta(\mu^m))_s) \right|^p ds \mid \mathcal{F}_t \right], \quad P\text{-a.s.} \quad (4.52)
\]

Then one can deduce from (2.9), (2.3), (2.4), (4.39), (4.43) and (4.44) that

\[
|Y_t^{\Theta^m} - Y_t^{\Theta^m}(\tau_m, \eta_m)|^p \leq c_0 E\left[ |Y_t^{\Theta^m}(\tau_m, \eta_m) - \eta_m|^p \mid \mathcal{F}_t \right]
\]

\[
\leq c_0 E\left[ \int_{\tau}^T \left( |X_{\tau_m}^{\Theta^m} - X_{\tau_m}^{\Theta^m}|^2 + |\eta_m|^p + |(\beta(\mu^m))_s|^2 \right) ds \mid \mathcal{F}_t \right]
\]

\[
\leq c_0 E\left[ \left( \eta_m - \tau \right) \sup_{s \in [t,T]} |X_{\tau_m}^{\Theta^m} - X_{\tau_m}^{\Theta^m}|^2 \right] + c_0 \left\{ (1 + (|x| + \delta)^2 + \left( C_{\delta, \phi}^0 + \frac{c_0}{m} \right) + \kappa^2 \right\}
\]

\[
\leq \frac{C(\kappa, \delta)^2}{m^2} + \frac{C(\kappa, \delta, \delta)}{m} + \frac{c_0}{m^{p+1}} \leq \frac{C(\kappa, \delta)}{m}, \quad P\text{-a.s.} \quad (4.53)
\]

Applying (2.10) with $(\zeta, \tau) = (\tau_m, \eta_m)$, applying (4.40) with $\eta = \eta_m$ and using (4.47) yield that $P$-a.s.

\[
Y_t^{\Theta^m}(\tau_m, \eta_m) = Y_t^{\Theta^m}(\tau, Y_t^{\Theta^m}(\tau_m, \eta_m)) \geq Y_t^{\Theta^m}(\tau, \eta_m) - \frac{C(\kappa, \delta)}{m^{1/p}}
\]

\[
= Y_t^{\Theta}(\tau, \eta_m) - \frac{C(\kappa, \delta, \delta)}{m^{1/p}} \geq Y_t^{\Theta}(\tau, \phi(\tau, X_{\tau}^T)) - \frac{C(\kappa, \delta)}{m^{1/p}}. \quad (4.54)
\]

As $\mu^m = \hat{\mu}$ on $[t, \tau_m]$, taking $(\tau, A) = (\tau_m, \emptyset)$ in Definition 2.2 shows that $\beta(\mu^m) = \beta(\hat{\mu})$, $ds \times dP$-a.s. on $[t, \tau_m]$, and then applying (2.7) with $(\tau, A) = (\tau_m, \emptyset)$ yields that $P$-a.s.

\[
X_{\tau_m}^{\Theta} = X_{\tau_m}^{\Theta}, \quad \forall s \in [t, \tau_m].
\]

(4.55)

Given $i = 1, \ldots, N_m$, (4.55) shows that $X_{i, \tau_m}^{\Theta} = X_{i, \tau_m}^{\Theta}$, $P$-a.s. on $\tilde{A}_{\tilde{m}} \setminus A_{\tilde{m}}$. As $\mu^m = \tilde{\mu}$ on $[t, \tau_m] \cup [\tau_m, T]_{\tilde{A}_{\tilde{m}} \setminus A_{\tilde{m}}}$, Definition 2.2 shows that $\beta(\mu^m) = \beta(\tilde{\mu})$, $ds \times dP$-a.s. on $[t, \tau_m] \cup [\tau_m, T]_{\tilde{A}_{\tilde{m}} \setminus A_{\tilde{m}}}$. So $((\mu^m)^i, [\beta(\mu^m)]^i) = ((\tilde{\mu})^i, [\beta(\tilde{\mu})]^i)$ holds $ds \times dP$-a.s. on $[\tau_m, T]_{\tilde{A}_{\tilde{m}} \setminus A_{\tilde{m}} = [t, T] \times (A_{\tilde{m}} \setminus A_{\tilde{m}})}$. Then by (4.28) and a similar argument to (4.45), it holds $P$-a.s. on $\tilde{A}_{\tilde{m}} \setminus A_{\tilde{m}}$ that

\[
Y_{\tau_m}^{\Theta}(T, g(X_{\tau_m}^{\Theta})) = Y_{\tau_m}^{\Theta}(T, g(X_{\tau_m}^{\Theta})) = J(\Theta^i) = J(\tilde{\Theta}^i) = Y_{\tau_m}^{\tilde{\Theta}}(T, g(X_{\tau_m}^{\tilde{\Theta}})) = Y_{\tau_m}^{\tilde{\Theta}}(T, g(X_{\tau_m}^{\tilde{\Theta}})),
\]

where $\tilde{\Theta}^i \triangleq (t_i, X_{t_i}^{\tilde{\Theta}}, [\beta(\tilde{\mu})]^i)$. Let $\tilde{\eta}_m \triangleq Y_{\tau_m}^{\Theta}(T, g(X_{\tau_m}^{\Theta})) \cap \tilde{\eta}_m \in L^p(\mathcal{F}_m)$ and set $\tilde{A}_m \triangleq \{ Y_{\tau_m}^{\Theta}(T, g(X_{\tau_m}^{\Theta})) < \tilde{\eta}_m \} \in \mathcal{F}_m$. Clearly, $\tilde{A}_m \subseteq A_{\tilde{m}}$, $P$-a.s. Applying (2.9) again, we can deduce from (4.44) and (4.56) that $P$-a.s.

\[
|Y_t^{\Theta^m}(\tau_m, \tilde{\eta}_m) - Y_t^{\Theta^m}(\tau_m, \eta_m)|^p \leq c_0 E\left[ |\tilde{\eta}_m - \eta_m|^p \mid \mathcal{F}_t \right] = c_0 E\left[ 1_{\tilde{A}_m} |\tilde{\eta}_m - \eta_m|^p + 1_{A_{\tilde{m}}} |Y_{\tau_m}^{\Theta^m}(T, g(X_{\tau_m}^{\Theta^m})) - \eta_m|^p \mid \mathcal{F}_t \right]
\]

\[
\leq c_0 E\left[ \left| X_{\tau_m}^{\Theta^m} - X_{\tau_m}^{\Theta^m} \right|^2 + \frac{1}{A_{\tilde{m}}} |Y_{\tau_m}^{\Theta^m}(T, g(X_{\tau_m}^{\Theta^m})) - \eta_m|^p \mid \mathcal{F}_t \right]
\]

\[
\leq \frac{C(\kappa, \delta, \delta)}{m} + c_0 E\left[ 1_{A_{\tilde{m}}} \left| Y_{\tau_m}^{\Theta^m}(T, g(X_{\tau_m}^{\Theta^m})) - \phi(\tau, X_{\tau_m}^{\Theta^m}) \right|^p \mid \mathcal{F}_t \right] + \frac{c_0}{m^p}
\]

\[
\leq \frac{C(\kappa, \delta, \delta)}{m} + c_0 E\left[ 1_{A_{\tilde{m}}} \left( \sup_{s \in [t,T]} |Y_{\tau_m}^{\Theta^m}(T, g(X_{\tau_m}^{\Theta^m}))|^p + (C_{\delta, \phi}^0)^p \right) \mid \mathcal{F}_t \right].
\]

(4.57)

Applying (2.10) with $(\zeta, \tau) = (\tau_m, T, g(X_{\tau_m}^{\Theta^m}))$, we see from Proposition 1.2 (2), (4.57) and (4.54) that $P$-a.s.

\[
Y_t^{\Theta^m}(T, g(X_{\tau_m}^{\Theta^m})) = Y_t^{\Theta^m}(\tau_m, Y_t^{\Theta^m}(T, g(X_{\tau_m}^{\Theta^m}))) \geq Y_t^{\Theta^m}(\tau_m, \tilde{\eta}_m)
\]

\[
\geq Y_t^{\Theta}(\tau, \phi(\tau, X_{\tau_m}^{\Theta^m})) - \frac{C(\kappa, \delta, \delta)}{m^{1/p}} - c_0 \left\{ E\left[ 1_{A_{\tilde{m}}} \sup_{s \in [t,T]} \left| Y_{\tau_m}^{\tilde{\Theta}}(T, g(X_{\tau_m}^{\tilde{\Theta}})) \right|^p + (C_{\delta, \phi}^0)^p \right] \mid \mathcal{F}_t \right\}^{1/2}.
\]

(4.58)
Letting \( \tilde{A}_m \overset{\Delta}{=} \left\{ E \left[ 1_{A_m} \left( \sup_{s \in [t,T]} \left| Y_{s}^{\Phi}(T, g(X_{T}^{\Phi})) \right|^{p} + (C_{x,\delta}^\phi)^{p} \right| \mathcal{F}_t \right] > 1/m \right\} \), one can deduce that

\[
P(\tilde{A}_m) \leq m E \left[ E \left[ 1_{A_m} \left( \sup_{s \in [t,T]} \left| Y_{s}^{\Phi}(T, g(X_{T}^{\Phi})) \right|^{p} + (C_{x,\delta}^\phi)^{p} \right| \mathcal{F}_t \right] \right] 
\leq \sum_{i=1}^{N} m E \left[ (A_{m,i}) \left( \sup_{s \in [t,T]} \left| Y_{s}^{\Phi}(T, g(X_{T}^{\Phi})) \right|^{p} + (C_{x,\delta}^\phi)^{p} \right) \right] \leq m^{-p}.
\]

Multiplying \( 1_{\tilde{A}_m} \) to both sides of (4.58) yields that

\[
1_{\tilde{A}_m} I(t, x, \beta) \geq 1_{\tilde{A}_m} J(t, x, \mu, \beta(\mu^{m})) \geq 1_{\tilde{A}_m} Y_{t}^{\Phi}(\tau, \phi(\tau, X_{\tau}^{\Phi})) - \frac{C(\kappa, x, \delta)}{m^{1/p}}, \quad P\text{-a.s.} \quad (4.59)
\]

As \( \sum_{m \in \mathbb{N}} P(\tilde{A}_m) \leq \sum_{m \in \mathbb{N}} m^{-p} < \infty \), Borel-Cantelli theorem shows that \( P(\lim_{m \to \infty} 1_{\tilde{A}_m} = 1) = 0 \). It follows that \( P(\lim_{m \to \infty} 1_{\tilde{A}_m} = 1) = 1 \) and thus

\[
\lim_{m \to \infty} 1_{\tilde{A}_m} = 0, \quad P\text{-a.s.} \quad (4.60)
\]

So letting \( m \to \infty \) in (4.59) yields that \( I(t, x, \beta) \geq Y_{t}^{l, x, \mu, \beta(\mu)}(\tau_{\beta, \mu}, \phi(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{l, x, \mu, \beta(\mu)})) \), \( P\)-a.s. Taking essential supremum over \( \mu \in \mathcal{U}_t \) and then taking essential infimum over \( \beta \in \mathcal{B}_t \), we obtain

\[
w_{1}(t, x) \geq \operatorname{essinf} \operatorname{esssup}_{\beta \in \mathcal{B}_t} Y_{t}^{l, x, \mu, \beta(\mu)} \left( \tau_{\beta, \mu}, \phi(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{l, x, \mu, \beta(\mu)}) \right), \quad P\text{-a.s.}
\]

1b) Now let us show the other side. Fix \( m \in \mathbb{N} \). For \( i = 1, \cdots, N_{m} \), (4.31) shows that there exists \( (A_{i}^{m}, \beta_{i}^{m}) \in \mathcal{F}_{1} \times \mathcal{B}_{t} \) with \( P(\mathcal{A}_{i}^{m}) \geq 1 - \frac{i+1}{1+p} N_{m}^{-1} \) such that

\[
\tilde{\phi}(t_{i}, x_{i}) \geq w_{1}(t_{i}, x_{i}) \geq I(t_{i}, x_{i}, \beta_{i}^{m}) - 1/m, \quad P\text{-a.s. on } \mathcal{A}_{i}^{m}. \quad (4.61)
\]

Let \( \beta_{\psi} \) be the \( \mathcal{B}_{t} \)-strategy considered in (4.10) and fix \( \beta \in \mathcal{B}_{t} \). For any \( \mu \in \mathcal{U}_{t} \), we simply denote \( \tau_{\beta, \mu} \) by \( \tau_{\mu} \) and define

\[
(\tilde{\beta}(\mu))_{s} \overset{\Delta}{=} 1_{\{ s < \tau_{\mu} \}} (\beta(\mu))_{s} + 1_{\{ s \geq \tau_{\mu} \}} (\beta_{\psi}(\mu))_{s}, \quad \forall \ s \in [t, T],
\]

which is a \( \mathcal{V}_{t} \)-control by Lemma 2.1. By (A-u), it holds \( ds \times dP \)-a.s. that

\[
[(\tilde{\beta}(\mu))_{s}]_{\mathcal{V}} = 1_{\{ s < \tau_{\mu} \}} [(\beta(\mu))_{s}]_{\mathcal{V}} + 1_{\{ s \geq \tau_{\mu} \}} [(\beta_{\psi}(\mu))_{s}]_{\mathcal{V}} \leq \kappa + (C_{x} \vee \kappa)[\mu_{s}]_{\mathcal{V}}. \quad (4.62)
\]

To see \( \tilde{\beta} \in \mathcal{B}_{t} \), we let \( \mu^{1}, \mu^{2} \in \mathcal{U}_{t} \) such that \( \mu^{1} = \mu^{2} \), \( ds \times dP \)-a.s. on \( [t, \tau \cup [\tau, T]_{A} \) for some \( \tau \in S_{t, T} \) and \( A \in \mathcal{F}_{\tau} \). Since \( \beta(\mu^{1}) = \beta(\mu^{2}) \), \( ds \times dP \)-a.s. on \( [t, \tau \cup [\tau, T]_{A} \) by Definition 2.2, it holds \( ds \times dP \)-a.s. on \( ([t, \tau \cup [t, T]_{A}) \cap [t, \tau_{\mu^{1}} \wedge \tau_{\mu^{2}}] \) that

\[
(\tilde{\beta}(\mu^{1}))_{s} = (\tilde{\beta}(\mu^{2}))_{s} = (\tilde{\beta}(\mu^{2}))_{s}. \quad (4.63)
\]

And (4.7) shows that except on a \( P\)-null set \( \mathcal{N} \)

\[
1_{A_{s}} X_{s}^{\Phi_{1}} + 1_{A_{s}} X_{s}^{\Phi_{2}} = 1_{A_{s}} X_{s}^{\Phi_{3}} + 1_{A_{s}} X_{s}^{\Phi_{4}}, \quad \forall \ s \in [t, T].
\]

Then it holds for any \( \omega \in A \cap \mathcal{N}^{c} \) that

\[
\tau_{\mu^{1}}(\omega) = \inf \left\{ s \in (t, T) : \left( X_{s}^{\Phi_{1}}(\omega) \right) \notin O_{\delta}(t, x) \right\} = \inf \left\{ s \in (t, T) : \left( X_{s}^{\Phi_{2}}(\omega) \right) \notin O_{\delta}(t, x) \right\} = \tau_{\mu^{2}}(\omega).
\]

Let \( A_{a} \overset{\Delta}{=} \{ \tau \geq \tau_{\mu^{1}} \wedge \tau_{\mu^{2}} \}. \) We can deduce from (4.64) that for any \( \omega \in A_{a} \cap \{ \tau_{\mu^{1}} \leq \tau_{\mu^{2}} \} \cap \mathcal{N}^{c} \)

\[
\tau_{\mu^{1}}(\omega) = \inf \left\{ s \in (t, T) : \left( X_{s}^{\Phi_{1}}(\omega) \right) \notin O_{\delta}(t, x) \right\} = \inf \left\{ s \in (t, \tau(\omega)) : \left( X_{s}^{\Phi_{1}}(\omega) \right) \notin O_{\delta}(t, x) \right\} \geq \inf \left\{ s \in (t, \tau(\omega)) : \left( X_{s}^{\Phi_{2}}(\omega) \right) \notin O_{\delta}(t, x) \right\} = \tau_{\mu^{2}}(\omega) \geq \tau_{\mu^{1}}(\omega).
\]
Similarly, it holds on $A_o \cap \{\tau_{\mu^2} \leq \tau_{\mu^1}\} \cap \mathcal{N}^c$ that $\tau_{\mu^1} = \tau_{\mu^2}$. So

$$\tau_{\mu^1} = \tau_{\mu^2} \quad \text{on } A \triangleq (A \cup A_o) \cap \mathcal{N}^c.$$  \hfill (4.65)

Since $[t, \tau \cap [\tau_{\mu^1} \wedge \tau_{\mu^2}, T] = [\tau_{\mu^1} \wedge \tau_{\mu^2}, \tau] \cap A_o$ and $[t, T]_A \cap [\tau_{\mu^1} \wedge \tau_{\mu^2}, T] = [\tau_{\mu^1} \wedge \tau_{\mu^2}, T]_A$, (4.65) leads to that

$$([t, \tau] \cup [t, T])_A \cap [\tau_{\mu^1} \wedge \tau_{\mu^2}, T]_A = [\tau_{\mu^1}, T]_A \cap [\tau_{\mu^2}, T]_A.$$  

Thus it holds $ds \times dP -$ a.s. on $([t, \tau] \cup [t, T])_A \cap [\tau_{\mu^1} \wedge \tau_{\mu^2}, T]$ that $(\hat{\beta}(\mu^1))_s = \psi(s, \mu_{s^2}) = (\hat{\beta}(\mu^2))_s$, which together with (4.66) shows that $\hat{\beta}$ is a $\mathcal{B}_t$-strategy.

Given $\mu \in \mathcal{U}_t$, we set $\Theta_\mu \triangleq (t, \mu, \beta(\mu))$ and $\hat{\Theta}_\mu \triangleq (t, \mu, \hat{\beta}(\mu))$. For $i = 1, \cdots, N_m$, analogous to $\hat{A}_t$ of part (1a), $A_{t}^{\mu, m} \triangleq \{(\tau_i, \Theta_{\mu}^t) \in \mathcal{D}_m(s_i, x_i) \cup \mathcal{D}_m(s_j, x_j)\}$ belongs to $\mathcal{F}_t \cap \mathcal{F}_s$. By the continuity of process $X_{\Theta_t}$, $(\tau_i, X_{\Theta_t}^{\mu, m}) \in \partial \mathcal{O}_s(x, t), \ P$-a.s. So $\{A_{t}^{\mu, m}\}_{i=1}^{N_m}$ forms a partition of $\mathcal{N}^c$ for some $P$-null set $\mathcal{N}_t$. Then we can define an $F$-stopping time $\tau_{\mu} \triangleq \sum_{i=1}^{N_m} 1_{A_{t}^{\mu, m}} t_i + 1_{\mathcal{N}_t} T \geq \tau_{\mu}$ as well as a process

$$(\beta_{\mu}(\mu))_s \triangleq 1_{(s+ \tau_{\mu})} (\hat{\beta}(\mu))_s + 1_{(s \geq \tau_{\mu})} \left( \sum_{i=1}^{N_m} 1_{A_{t}^{\mu, m} \cap A_{t}^{\mu} (\beta_{\mu}^{1+} \{[\mu]_t^m\})_s + 1_{A_{t}^{\mu}} (\hat{\beta}(\mu))_s \right)$$  

$$= 1_{A_t^{\mu} (\hat{\beta}(\mu))_s} + \sum_{i=1}^{N_m} 1_{A_{t}^{\mu, m} \cap A_{t}^{\mu} (1_{(s < t_i)} (\hat{\beta}(\mu))_s + 1_{(s \geq t_i)} (\beta_{\mu}^{1+} \{[\mu]_t^m\})_s), \ \forall s \in [t, T],$$  \hfill (4.66)

where $A_{t}^{\mu} = \left( \bigcup_{i=1}^{N_m} (A_{t}^{\mu, m} \cap A_{t}^{\mu}) \right) \cap \mathcal{N}_t$.

We claim that $\beta_{\mu}$ is a $\mathcal{B}_t$-strategy. Using a similar argument to that in part (1a) for the measurability of the pasted control $\mu_{\mu}$, one can deduce that the process $\beta_{\mu}(\mu)$ is $F-$progressively measurable. For $i = 1, \cdots, N_m$, let $C_{i}^{\mu, m} > 0$ be the constant associated to $\beta_{\mu}^{1+}$ in Definition 2.2 (i). Setting $C_{\mu} = C_{\beta} \lor \kappa \lor \max\{C_{i}^{t} : i = 1, \cdots, N_m\}$, we can deduce from (4.66) and (A-u) that $ds \times dP -$ a.s.

$$[ (\beta_{\mu}(\mu)\mu)]_s^v = 1_{(s < \tau_{\mu})} \left( (\hat{\beta}(\mu))_s^v + 1_{(s \geq \tau_{\mu})} \left( \sum_{i=1}^{N_m} 1_{A_{t}^{\mu, m} \cap A_{t}^{\mu} (\beta_{\mu}^{1+} \{[\mu]_t^m\})_s^v + 1_{A_{t}^{\mu}} (\hat{\beta}(\mu))_s^v \right) \right)$$  

$$\leq (1_{(s < \tau_{\mu})} + 1_{(s \geq \tau_{\mu})})_s^v \left( (\beta_{\mu}(\mu))_s^v + 1_{A_{t}^{\mu} \cap A_{t}^{\mu} (\kappa + C_{\mu}^{\mu} \{[\mu]_t^m\})_s^v \right) \leq \kappa + C_{\mu}[\mu]_s^v. \ \ (4.67)$$

Let $E \int_{t}^{T} [\mu]_s^v ds < \infty$ for some $q > 2$. It follows from (4.67) that

$$E \int_{t}^{T} [\beta_{\mu}(\mu)]_s^q ds \leq 2^{q-1} \kappa^q T + 2^{q-1} C_{\mu}^q E \int_{t}^{T} [\mu]_s^q ds \leq \infty.$$  

Hence $\beta_{\mu}(\mu) \in \mathcal{V}_t$.

Let $\mu^1, \mu^2 \in \mathcal{U}_t$ such that $\mu^1 = \mu^2$, $ds \times dP -$ a.s. on $[t, \tau \cap [\tau_{\mu^1} \wedge \tau_{\mu^2}, T]]_A$ for some $\tau \in \mathcal{S}_t$ and $A \in \mathcal{F}_T$. As $\hat{\beta}(\mu^1) = \hat{\beta}(\mu^2)$, $ds \times dP -$ a.s. on $[t, \tau] \cap [\tau_{\mu^1} \wedge \tau_{\mu^2}, T]_A$ by Definition 2.2 it holds $ds \times dP -$ a.s. on $[t, \tau] \cap [\tau_{\mu^1} \wedge \tau_{\mu^2}, T]_A$ that

$$\beta_{\mu^1}(\mu^1) = \beta_{\mu^2}(\mu^2) = \beta_{\mu^1}(\mu^2). \ \ (4.68)$$

Definition 2.2 also shows that $(\mu^1, \beta_{\mu^1})(\mu^2, \beta_{\mu^2})$, $ds \times dP -$ a.s. on $[t, \tau] \cap [\tau_{\mu^1} \wedge \tau_{\mu^2}, T]_A$. Similar to part (1a), we again have (4.64) except on a $P$-null set $\mathcal{N}^c$, and (4.65) still holds on $A \triangleq (A \cup A_o) \cap \mathcal{N}^c$ with $A_o = \{\tau_{\mu^1} \wedge \tau_{\mu^2}\}$. Plugging (4.65) into (4.64) yields that

$$X_{\tau_{\mu^1}}^{\Theta_{\mu^1}} = X_{\tau_{\mu^2}}^{\Theta_{\mu^2}} \text{ holds on } A.$$  \hfill (4.69)

Given $i = 1, \cdots, N_m$, since it holds $ds \times dP -$ a.s. on $[t, \tau] \cap [\tau_{\mu^1} \wedge \tau_{\mu^2}, T]_A \cap ([t, T] \times \Omega)$ that $([\mu^1]_{t_i})_s = [\mu^2]_{t_i}$, taking $(\tau, A) = (\tau \lor t_i, A)$ in Definition 2.2 with respect to $\beta_{\mu}^{1+}$ yields that for $ds \times dP -$ a.s. $(s, \omega) \in [t_i, \tau \lor t_i \cap [\tau \lor t_i, T]_A \cap ([t_i, T] \times \Omega)$

$$\beta_{\mu}^{1+}([\mu^1]_{t_i})_s(\omega) = \beta_{\mu}^{1+}([\mu^2]_{t_i})_s(\omega).$$  \hfill (4.70)
Given \( \omega \in A_\kappa \triangleq A \cap \mathcal{A}_t^{1,m} \), (4.65) and (4.69) imply that
\[
\left( \tau_{\mu_1}(\omega), X_{\tau_{\mu_1}}^\Theta_{t_1}(\omega) \right) = \left( \tau_{\mu_1}(\omega), X_{\tau_{\mu_1}}^\Theta_{t_1}(\omega) \right) \in \mathcal{D}_m(s_i, x_i) \cup \mathcal{D}_m(s_j, x_j), \quad \text{i.e., } \omega \in \mathcal{A}_t^{\mu_1,m}.
\]

So \( A_\kappa \subset \mathcal{A}_t^{1,m} \cap \mathcal{A}_t^{2,m} \), and it follows that \( 1_{A_\kappa} \tau_{\mu_1}^m = 1_{A_\kappa} t_i = 1_{A_\kappa} \tau_{\mu_2}^m \). Then one can deduce that
\[
\left( [t, \tau \cup [t, T]_{A}] \cap [\tau_{\mu_1}^m \wedge \tau_{\mu_2}^m, T]_{A \cap A_\kappa} \right) = \left( [t, \tau \cup [t, T]_{A}] \cap ([t, T] \cap (A_\kappa \cap A_{\kappa}^\gamma)) \right) \subset \left( [t, T] \times \mathcal{A}_t^{\mu_1,m} \cap \mathcal{A}_t^{\mu_2,m} \cap \mathcal{A}_t^{\gamma,m} \right),
\]
which together with (4.70) shows that for \( ds \times dP \)-a.s. \( s, \omega \in \left( [t, \tau \cup [t, T]_{A}] \cap [\tau_{\mu_1}^m \wedge \tau_{\mu_2}^m, T]_{A \cap A_\kappa} \right) \subset \left( A_\kappa \cap A_{\kappa} \cap A_{\kappa}^\gamma \right) \),
\[
\left( \beta_m(\mu^1) \right)_s(\omega) = \left( \beta_m^1(\mu^1) \right)_s(\omega) = \left( \beta_m^1(\mu^2) \right)_s(\omega) = \left( \beta_m(\mu^2) \right)_s(\omega).
\]

Analogous to (4.71), \( \left( [t, \tau \cup [t, T]_{A}] \cap [\tau_{\mu_1}^m \wedge \tau_{\mu_2}^m, T]_{A \cap A_\kappa} \right) \subset \left( A_\kappa \cap A_{\kappa} \cap A_{\kappa}^\gamma \right) \cap \left( A_\kappa \cap A_{\kappa} \cap A_{\kappa}^\gamma \right) \)
\[
\left( \beta_m(\mu^1) \right)_s(\omega) = \left( \beta_m(\mu^2) \right)_s(\omega), \quad \text{ds} \times \text{dP} \text{-a.s. on } ([t, \tau \cup [t, T]_{A}] \cap [\tau_{\mu_1}^m \wedge \tau_{\mu_2}^m, T]_{A \cup A_\kappa}).
\]

As \( [\tau_{\mu_1}^m \wedge \tau_{\mu_2}^m, T]_{A \cap A_\kappa} \subset [\tau_{\mu_1} \wedge \tau_{\mu_2}, T]_{A \cap A_\kappa} \subset [\tau, T]_{A \cap A_\kappa} \subset [\tau, T]_{A \cap A_\kappa} \), one can deduce that \( ([t, \tau \cup [t, T]_{A}] \cap [\tau_{\mu_1}^m \wedge \tau_{\mu_2}^m, T]_{A \cup A_\kappa}) \cap \left( A_\kappa \cap A_{\kappa} \cap A_{\kappa}^\gamma \right) \).

Next, let \( \mu \in \mathcal{U}_t \) and set \( \Theta_{\mu} \equiv (t, x, \mu, \beta_{\mu}(\mu)) \). As \( \beta_{\mu}(\mu) = \widehat{\beta}(\mu) = \beta_{\mu}(\mu) \) on \( \tau_{\mu} \), taking \( (\tau, A) = (\tau_{\mu}, \emptyset) \) in (2.7) yields that \( P \)-a.s.
\[
X_{s}^{\Theta_{\mu}} = X_{s_{\mu}}^{\Theta_{\mu}} = X_{s_{\mu}}^{\Theta_{\mu}} \in \mathcal{O}_x(d), \quad \forall s \in [t, \tau_{\mu}].
\]

Thus, for any \( \eta \in L^p(F_{\tau_{\mu}}) \), the BSDE \( \left( t, \eta, f_{\tau_{\mu}}^{\Theta_{\mu}} \right) \) and the BSDE \( \left( t, \eta, f_{\tau_{\mu}}^{\Theta_{\mu}} \right) \) are essentially the same. To wit,
\[
\left( Y_{s}^{\Theta_{\mu}}(\tau_{\mu}, \eta), Z_{s}^{\Theta_{\mu}}(\tau_{\mu}, \eta) \right) = \left( Y_{s}^{\Theta_{\mu}}(\tau_{\mu}, \eta), Z_{s}^{\Theta_{\mu}}(\tau_{\mu}, \eta) \right).
\]

Given \( A \in \mathcal{F}_t \), similar to (4.41), we can deduce from (4.74) that
\[
1_{A} \sup_{r \in [t, s]} \left| X_{r}^{\Theta_{\mu}} - X_{r}^{\Theta_{\mu}} \right| \leq \int_{\tau_{\mu}^m \wedge s}^{s} 1_{A} \left| b(r, X_{r}^{\Theta_{\mu}}(r, \mu_r), \psi(r, \mu_r)) \right| dr
\]
\[
\quad + \int_{\tau_{\mu}^m \wedge s}^{s} 1_{A} \sigma(r, X_{r}^{\Theta_{\mu}}(r, \mu_r), \psi(r, \mu_r)) d\mathbb{B}(r), \quad s \in [t, T],
\]

where we used the fact that \( \beta_{\mu}(\mu) = \widehat{\beta}(\mu) = \beta_{\mu}(\mu) \) on \( \tau_{\mu}, \tau_{\mu}^m \). Let \( \mathcal{C}(\kappa, x, \delta) \) denote a generic constant, depending on \( \kappa + |x| + \delta, C_{x, \delta} \triangleq \sup \left\{ \varphi(s, x) : (s, x) \in \mathcal{O}_x(t, \tau_{\mu} \cap (\mathbb{R}) \cap \mathbb{R}^+ \right\} \), \( T, \gamma, p \) and \( |g(0)| \), whose form may vary from line to line. Since \( \tau_{\mu}^m - \tau_{\mu} \leq \sum_{i=1}^{N_A} 1_{A_{\mu_1} \wedge \mu_2} 2\delta_{\mu_1} \leq \frac{C m}{m}, P \)-a.s., using similar arguments to those that lead to (4.42) and using an analogous decomposition and estimation to (4.11), we can deduce that
\[
E \left[ 1_{A} \sup_{r \in [t, s]} \left| X_{r}^{\Theta_{\mu}}(r, \mu_r) - X_{r}^{\Theta_{\mu}}(r, \mu_r) \right|^2 \right]
\]
\[
\leq 4E \int_{\tau_{\mu}^m \wedge s}^{s} 1_{A} \left| b(r, X_{r}^{\Theta_{\mu}}(r, \mu_r), \psi(r, \mu_r)) \right|^2 dr + 8E \int_{\tau_{\mu}^m \wedge s}^{s} 1_{A} \left| \sigma(r, X_{r}^{\Theta_{\mu}}(r, \mu_r), \psi(r, \mu_r)) \right|^2 dr
\]
\[
\leq 24\gamma^2 \int_{t}^{s} E \left[ 1_{A} \sup_{r \in [t, r]} \left| X_{r}^{\Theta_{\mu}}(r, \mu_r) - X_{r}^{\Theta_{\mu}}(r, \mu_r) \right|^2 \right] dr + \frac{\mathcal{C}(\kappa, x, \delta)}{m} P(A), \quad \forall s \in [t, T].
\]

Then similar to (4.44), an application of Gronwall’s inequality leads to that
\[
E \left[ \sup_{r \in [t, T]} \left| X_{r}^{\Theta_{\mu}}(r, \mu_r) - X_{r}^{\Theta_{\mu}}(r, \mu_r) \right|^2 \mid F_t \right] \leq \frac{\mathcal{C}(\kappa, x, \delta)}{m}, \quad P \text{-a.s.}
\]

(4.76)
Let $i = 1, \cdots, N_m$ and set $\Theta^{m,t_i}_\mu \overset{\Delta}{=} \left( t_i, X^{\Theta^m}_t, [\mu]^t_i, (\beta_m(\mu))^t_i \right)$. Similar to (4.45), it holds $P$-a.s. that
\begin{equation}
Y^{\Theta^m}(t, g(X^{\Theta^m}_t)) = J(\Theta^{m,t_i}). \tag{4.77}
\end{equation}

Since $[\beta_m(\mu)]^t_i(\omega) = (\beta_m(\mu))^t_i(\omega)$ for any $(r, \omega) \in [t_i, T] \times (A^{\mu,m}_m \cap A^m_m)$, one can deduce from (4.77), (4.28) and (2.13) that it holds $P$-a.s. on $A^{\mu,m}_m \cap A^m_m \in F_t$ that
\begin{equation}
Y^{\Theta^m}(t, g(X^{\Theta^m}_t)) = Y^{\Theta^m}_t(t, g(X^{\Theta^m}_t)) = J(t_i, X^{\Theta^m}_t, [\mu]^t_i, (\beta_m(\mu))^t_i) \leq J(t_i, X^{\Theta^m}_t, [\mu]^t_i, (\beta_m(\mu))^t_i) + c_0 \left| X^{\Theta^m}_t - X^{\Theta^m}_t \right|^{2/p}.
\end{equation}

As $|X^{\Theta^m}_t - x|^q < (\delta^m)^{q/p} < \epsilon^{-2}/p \leq 1/m$ on $A^{\mu,m}_m$, we can also deduce from (2.13), (4.61), (4.35) and the continuity of $\phi$ that it holds $P$-a.s. on $A^{\mu,m}_m \cap A^m_m$ that
\begin{equation}
J(t_i, X^{\Theta^m}_t, [\mu]^t_i, (\beta_m(\mu))^t_i) \leq J(t_i, x, [\mu]^t_i, (\beta_m(\mu))^t_i) + \frac{c_0}{m} \leq \bar{I}(t_i, x, (\beta_m(\mu))^t_i) + \frac{c_0}{m} \leq \bar{\phi}(t_i, x) + \frac{c_0}{m} \equiv \tilde{\eta}_m \in L^\infty(F_{t_i}).
\end{equation}

Thus it holds $P$-a.s. on $\cup_{i=1}^N (A^{\mu,m}_m \cap A^m_m)$ that
\begin{equation}
Y^{\Theta^m}_t(t, g(X^{\Theta^m}_t)) \leq \bar{\eta}_m + c_0 \left| X^{\Theta^m}_t - X^{\Theta^m}_t \right|^{2/p} \equiv \tilde{\eta}_m \in L^p(F_{t_i}).
\end{equation}

By (4.39), it holds $P$-a.s. that
\begin{equation}
\left| Y^{\Theta^m}_t(\tau_m, \eta^m_m) - Y^{\Theta^m}_t(\tau_m, \bar{\phi}(\tau_m, X^{\Theta^m}_t)) \right| \leq c_0 \left( \left| \eta^m_m - \bar{\phi}(\tau_m, X^{\Theta^m}_t) \right| \right| F_t \right| \tag{4.78}
\end{equation}

Similar to (4.52), one can deduce that
\begin{equation}
E \left[ Y^{\Theta^m}_t(\tau_m, \eta^m_m) - Y^{\Theta^m}_t(\tau_m, \bar{\phi}(\tau_m, X^{\Theta^m}_t)) \right| F_t \right| \leq c_0 E \left[ \left| \eta^m_m - \bar{\phi}(\tau_m, X^{\Theta^m}_t) \right| \right| F_t \right| \quad P\text{-a.s.}
\end{equation}

Using an analogous decomposition and estimation to (4.11), similar to (4.53), we can deduce from (4.76) that
\begin{equation}
\left| Y^{\Theta^m}_t(\tau_m, Y^{\Theta^m}_t(\tau_m, \eta^m_m)) - Y^{\Theta^m}_t(\tau_m, \eta^m_m) \right| \leq E \left[ \left| \eta^m_m - \bar{\phi}(\tau_m, X^{\Theta^m}_t) \right| \right| F_t \right| \leq \frac{\bar{C}(\kappa, x, \delta)}{m}, \quad P\text{-a.s.}
\end{equation}

Applying (2.10) with $(\zeta, \tau, \eta) = (\tau_m, \mu^m_m, \eta^m_m)$, applying (4.75) with $\eta = \eta^m_m$ and using (4.78) yield that $P$-a.s.
\begin{equation}
\left| Y^{\Theta^m}_t(\tau_m, \eta^m_m) - Y^{\Theta^m}_t(\tau_m, \bar{\phi}(\tau_m, X^{\Theta^m}_t)) \right| \leq \frac{C(\kappa, x, \delta)}{m^{1/p}} = \frac{C(\kappa, x, \delta)}{m^{1/p}} \leq \frac{C(\kappa, x, \delta)}{m^{1/p}} \leq \frac{C(\kappa, x, \delta)}{m^{1/p}}.
\end{equation}

As $\bar{\beta}_m(\mu) = \bar{\beta}(\mu)$, $dx \times dP$-a.s. on $[t, \tau_m^m]$, applying (2.7) with $(\tau, A) = (\tau_m^m, \emptyset)$ yields that $P$-a.s.
\begin{equation}
X^{\Theta^m}_s = X^{\Theta^m}_s, \quad \forall s \in [t, \tau_m^m]. \tag{4.80}
\end{equation}

Given $i = 1, \cdots, N_m$, (4.80) shows that $X^{\Theta^m}_t = X^{\Theta^m}_t$, $P$-a.s. on $A^{\mu,m}_m \cap A^m_m$. Since $[\beta_m(\mu)]^t_i(\omega) = (\beta_m(\mu))^t_i(\omega)$ for any $(r, \omega) \in [t_i, T] \times (A^{\mu,m}_m \cap A^m_m)$, one can deduce from (4.77) and (4.28) that it holds $P$-a.s. on $A^{\mu,m}_m \cap A^m_m$ that
\begin{equation}
Y^{\Theta^m}_t(T, g(X^{\Theta^m}_T)) = Y^{\Theta^m}_t(T, g(X^{\Theta^m}_T)) = J(\Theta^{m,t_i}) = J(\Theta^{m,t_i}) = Y^{\Theta^m}_T(T, g(X^{\Theta^m}_T)) = Y^{\Theta^m}_T(T, g(X^{\Theta^m}_T)). \tag{4.81}
\end{equation}
where $\Theta^\mu_t \triangleq \langle t, \Delta_{t^2}, \mu_t, \beta(\mu)_t \rangle$.

Given $A \in \mathcal{F}_t$, one can deduce that

$$1_A X_{\tau_\nu}^\mu = 1_A X_{\tau_\nu}^\mu + 1_A \int_t^{\tau_\nu} b\left(r, X_r^\mu, \mu_r, \beta(\mu)_r \right) dr + 1_A \int_t^{\tau_\nu} \sigma\left(r, X_r^\mu, \mu_r, \beta(\mu)_r \right) dB_r,$$

$$= 1_A X_{\tau_\nu}^\mu + \int_t^s 1_{(r \geq \tau_\nu)} 1_A b\left(r, X_r^\mu, \mu_r, \psi(r, \mu_r) \right) dr + \int_t^s 1_{(r \geq \tau_\nu)} 1_A \sigma\left(r, X_r^\mu, \mu_r, \psi(r, \mu_r) \right) dB_r, \quad s \in [t, T].$$

It then follows from (4.74) that

$$1_A \sup_{r \in [t, s]} |X_{\tau_\nu}^\mu| \leq 1_A(|x| + \delta) + \int_t^s 1_{(r \geq \tau_\nu)} 1_A \left|b\left(r, X_r^\mu, \mu_r, \psi(r, \mu_r) \right) \right| dr$$

$$+ \sup_{r \in [t, s]} \int_t^r 1_{(r \geq \tau_\nu)} 1_A \sigma\left(r, X_r^\mu, \mu_r, \psi(r, \mu_r) \right) dB_r, \quad s \in [t, T].$$

Using an analogous decomposition and estimation to (4.11), one can deduce from Hölder’s inequality, Doob’s martingale inequality, (2.1), (2.2), (4.74) and Fubini’s Theorem that

$$E\left[1_A \sup_{r \in [t, s]} |X_{\tau_\nu}^\mu|^2 \right] \leq \tilde{C}(\mu, \nu, \alpha) P(A) + c_0 E\left[\int_t^s 1_{(r \geq \tau_\nu)} 1_A \left|b\left(r, X_r^\mu, \mu_r, \psi(r, \mu_r) \right) \right|^2 \right] dr$$

$$\leq \tilde{C}(\mu, \nu, \alpha) c_0 P(A) + c_0 \int_t^s E\left[1_A \sup_{r \in [t, r']} \left|X_{\tau_\nu}^\mu\right|^2 \right] dr, \quad \forall s \in [t, T].$$

Then an application of Gronwall’s inequality shows that

$$E\left[1_A \sup_{r \in [t, s]} |X_{\tau_\nu}^\mu|^2 \right] \leq \tilde{C}(\mu, \nu, \alpha) e^{c_0(s-t)} , \quad s \in [t, T].$$

In particular,

$$E\left[1_A \sup_{r \in [t, T]} |X_{\tau_\nu}^\mu|^2 \right] = E\left[1_A \sup_{r \in [t, T]} |X_{\tau_\nu}^\mu|^2 \right] \leq \tilde{C}(\mu, \nu, \alpha) P(A).$$

Letting $A$ vary in $\mathcal{F}_t$ yields that

$$E\left[\sup_{r \in [t, T]} |X_{\tau_\nu}^\mu|^2 \right] \leq \tilde{C}(\mu, \nu, \alpha), \quad P\text{-a.s.} \quad (4.82)$$

Let $(\hat{\nu}, \hat{Z}) \in \mathcal{G}_{\mu, \nu}^\mu([t, T])$ be the unique solution of the following BSDE with zero generator:

$$\hat{Y}_t^\mu = Y_t^\mu \left(T, g(X_T^\mu) \right) - \int_s^T \hat{Z}_r^\mu dB_r, \quad s \in [t, T].$$

Analogous to (4.51), $(\hat{\nu}, \hat{Z}) \triangleq \left\{ (1_{s < \tau_\nu} \hat{Y}_s^\mu + 1_{s \geq \tau_\nu} Y_s^\mu \left(T, g(X_T^\mu) \right), 1_{s < \tau_\nu} \hat{Z}_s^\mu + 1_{s \geq \tau_\nu} Z_s^\mu \left(T, g(X_T^\mu) \right) \right\}_{s \in [t, T]} \in \mathcal{G}_{\mu, \nu}^\mu([t, T])$ solves the following BSDE

$$\hat{Y}_s^\mu = g(X_T^\mu) + \int_s^T 1_{(r \geq \tau_\nu)} \hat{Y}_r^\mu \left(r, \hat{Y}_r^\mu, \hat{Z}_r^\mu \right) dr - \int_s^T \hat{Z}_r^\mu dB_r, \quad s \in [0, T].$$

Then (2.8), (1.4) and Hölder’s inequality imply that $P\text{-a.s.}$

$$E\left[\sup_{s \in [\tau_\nu, T]} |Y_s^\mu \left(T, g(X_T^\mu) \right)|^p \right] \leq E\left[\sup_{s \in [t, T]} |\hat{Y}_s^\mu|^p \right] \leq c_0 E\left[g(X_T^\mu)^p \right] + \int_{\tau_\nu}^T f_{\mu}^\mu(s, 0, 0)^p ds \left[\mathcal{F}_t \right]$$

$$= c_0 E\left[g(X_T^\mu)^p \right] + \int_{\tau_\nu}^T f(s, X_s^\mu, 0, 0, \psi(s, \mu_s))^p ds \left[\mathcal{F}_t \right].$$

Using an analogous decomposition and estimation to (4.11), we can then deduce from (2.3), (2.4) and (4.82) that

$$E\left[\sup_{s \in [\tau_\nu, T]} |Y_s^\mu \left(T, g(X_T^\mu) \right)|^p \right] \leq c_0 + c_0 E\left[\sup_{s \in [\tau_\nu, T]} |X_s^\mu|^2 \right] \leq \tilde{C}(\mu, \nu, \alpha), \quad P\text{-a.s.} \quad (4.83)$$
4. Proofs

Let \( \hat{\mu}_m \triangleq Y_{\tau_m}^{\Theta_m}(T, g(X_{\tau_m}^{\Theta_m})) \) and set \( \bar{A}_m \triangleq \{ Y_{\tau_m}^{\Theta_m}(T, g(X_{\tau_m}^{\Theta_m})) > \hat{\mu}_m \} \) \( \in \mathcal{F}_{\tau_m}^{\Theta_m} \). Clearly, \( 1_{\bar{A}_m} \leq 1_{A_m} \), \( P \)-a.s. Applying (2.9) with \( \bar{p} = \frac{1}{m} \), we can deduce from Hölder's inequality, (4.76) and (4.81) that

\[
\left| Y_{\tau_m}^{\Theta_m}(\hat{\mu}_m) - Y_{\tau_m}^{\Theta_m}(\bar{\eta}_m) \right|^{\frac{2}{p}} \leq c_0 E \left[ |\hat{\eta}_m - \bar{\eta}_m|^p \right]^{\frac{2}{p}} + c_0 \left\{ E \left[ 1_{\bar{A}_m} \right] \right\}^{\frac{p-2}{p}} \left\{ E \left[ Y_{\tau_m}^{\Theta_m}(T, g(X_{\tau_m}^{\Theta_m})) - \bar{\eta}_m \right]^p \right\}^{\frac{2}{p}}.
\]

Applying (2.10) with \( (\zeta, \tau, \eta) = (\tau_m, T, g(X_{\tau_m}^{\Theta_m})) \), we see from Proposition 1.2 (2), (4.83) and (4.79) that

\[
P(\bar{A}_m) \leq m^{\frac{1}{p-m}} E \left[ \left\{ 1_{\cup_{i=1}^N (A_i)^c} \right\}^p \right] = m^{\frac{1}{p-m}} P( \cup_{i=1}^N (A_i)^c ) \leq m^{\frac{1}{p-m}} \sum_{i=1}^N P((A_i)^c) \leq m^{-p}.
\]

Multiplying \( 1_{\bar{A}_m} \) to both sides of (4.84) yields that

\[
1_{\bar{A}_m} \cdot J(t, x, \beta_m) \leq 1_{\bar{A}_m} \text{esssup}_{\mu \in U_t} Y_{t,x,\mu,\beta_m}(\tau_{\beta,\mu}, \tau_{\beta,\mu}, X_{t,x,\mu,\beta_m}(\tau)) + \frac{C(\tau, x, \delta)}{m^{1/p}}, \quad P\text{-a.s.}
\]

Since \( \bar{A}_m \) does not depend on \( \mu \) nor on \( \beta \), taking essential supremum over \( \mu \in U_t \) and applying Lemma 2.4 (2) yield that

\[
1_{\bar{A}_m} w_1(t, x) \leq 1_{\bar{A}_m} I(t, x, \beta_m) \leq 1_{\bar{A}_m} \text{esssup}_{\mu \in U_t} Y_{t,x,\mu,\beta_m}(\tau_{\beta,\mu}, \tau_{\beta,\mu}, X_{t,x,\mu,\beta_m}(\tau)) + \frac{C(\tau, x, \delta)}{m^{1/p}}, \quad P\text{-a.s.}
\]

Then taking essential infimum over \( \beta \in \mathcal{B}_t \) and using Lemma 2.4 (2) again, we obtain

\[
1_{\bar{A}_m} w_1(t, x) \leq \text{essinf}_{\beta \in \mathcal{B}_t} \text{esssup}_{\mu \in U_t} Y_{t,x,\mu,\beta_m}(\tau_{\beta,\mu}, \tau_{\beta,\mu}, X_{t,x,\mu,\beta_m}(\tau)) + \frac{C(\tau, x, \delta)}{m^{1/p}}, \quad P\text{-a.s.}
\]

As \( \sum_{m \in \mathbb{N}} P(\bar{A}_m) \leq \sum_{m \in \mathbb{N}} m^{-p} < \infty \), similar to (4.60), Borel-Cantelli theorem implies that \( \lim_{m \to \infty} 1_{\bar{A}_m} = 0 \), \( P \)-a.s. Thus, letting \( m \to \infty \) in (4.85) yields that

\[
w_1(t, x) \leq \text{essinf}_{\beta \in \mathcal{B}_t} \text{esssup}_{\mu \in U_t} Y_{t,x,\mu,\beta_m}(\tau_{\beta,\mu}, \tau_{\beta,\mu}, X_{t,x,\mu,\beta_m}(\tau)) , \quad P\text{-a.s.}
\]

2) For any \((t, x, y, z, u, v) \in [0, T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^d \times U \times V\), we define

\[
g(x) \triangleq -g(x) \quad \text{and} \quad f(t, x, y, z, u, v) \triangleq -f(t, x, y, z, u, v).
\]
Given \((\mu, \nu) \in \mathcal{U} \times \mathcal{V}_t\), we let \(\Theta\) stand for \((t, x, \mu, \nu)\). For any \(\tau \in \mathcal{S}_{t,T}\) and any \(\eta \in \mathbb{L}^2(\mathcal{F}_\tau)\), let \((Y^\Theta(\tau, \eta), Z^\Theta(\tau, \eta))\) denote the unique solution of the BSDE \((t, \eta, f^\Theta_t)\) in \(\mathbb{G}^\Theta_\mathcal{F}([t, T])\), where

\[
 f^\Theta_t(s, \omega, y, z) \overset{\triangle}{=} 1_{\{s < \tau(\omega)\}} \mathbb{I} \left( \left( s, X^\nu_t(\omega), y, z, \mu_s(\omega), \nu_s(\omega) \right) \right), \quad \forall \ (s, \omega, y, z) \in [t, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d.
\]

Multiplying \(-1\) in the BSDE \((t, \eta, f^\Theta_t)\) shows that \((-Y^\Theta(\tau, \eta), -Z^\Theta(\tau, \eta)) \in \mathbb{G}^\Theta_\mathcal{F}([t, T])\) solves the BSDE \((-t, -\eta, f^\Theta_t)\). To wit

\[
 (-Y^\Theta(\tau, \eta), -Z^\Theta(\tau, \eta)) = (Y^\Theta(\tau, -\eta), Z^\Theta(\tau, -\eta)). \tag{4.86}
\]

Given \((t, x) \in [0, T] \times \mathbb{R}^k\), let us consider the situation where player II acts first by choosing a \(\mathcal{V}_t\)-control to maximize \(\mathcal{Y}^t_{d, x, \alpha(\nu), \nu}(T, g(X^t_{d, x, \alpha(\nu), \nu}))\), where \(\alpha \in \mathfrak{A}_t\) is player I’s strategic response. The corresponding priority value of player II is \(w_2(t, x) \overset{\triangle}{=} \text{essinf}_{\alpha \in \mathfrak{A}_t} \text{essinf}_{\nu \in \mathcal{V}_t} \mathcal{Y}^t_{d, x, \alpha(\nu), \nu}(T, g(X^t_{d, x, \alpha(\nu), \nu}))\). We see from (4.86) that

\[
-w_2(t, x) = \text{esssup}_{\alpha \in \mathfrak{A}_t} \text{essinf}_{\nu \in \mathcal{V}_t} - \mathcal{Y}^t_{d, x, \alpha(\nu), \nu}(T, g(X^t_{d, x, \alpha(\nu), \nu})) = \text{esssup}_{\alpha \in \mathfrak{A}_t} \text{essinf}_{\nu \in \mathcal{V}_t} \mathcal{Y}^t_{d, x, \alpha(\nu), \nu}(T, g(X^t_{d, x, \alpha(\nu), \nu})) = w_2(t, x).
\]

Let \(t \in (0, T]\) and let \(\phi, \bar{\phi} : [t, T] \times \mathbb{R}^k \to \mathbb{R}\) be two continuous functions satisfying \(\phi(s, x) \leq w_2(s, x) \leq \bar{\phi}(s, x), (s, x) \in [t, T] \times \mathbb{R}^k\). As \(-\bar{\phi}(s, x) \leq w_2(t, x) \leq -\phi(s, x), (s, x) \in [t, T] \times \mathbb{R}^k\), applying the weak dynamic programming principle of part (1) yields that for any \(x \in \mathbb{R}^k\) and \(\delta \in (0, T-t]\)

\[
\text{essinf}_{\alpha \in \mathfrak{A}_t} \text{essinf}_{\nu \in \mathcal{V}_t} \mathcal{Y}^t_{d, x, \alpha(\nu), \nu}(\tau_{t, \nu}, -\bar{\phi}(\tau_{t, \nu}, X^t_{\tau_{t, \nu}, \alpha(\nu), \nu})) \leq w_2(t, x) \leq \text{essinf}_{\alpha \in \mathfrak{A}_t} \text{essinf}_{\nu \in \mathcal{V}_t} \mathcal{Y}^t_{d, x, \alpha(\nu), \nu}(\tau_{t, \nu}, -\phi(\tau_{t, \nu}, X^t_{\tau_{t, \nu}, \alpha(\nu), \nu})), \quad P-a.s.
\]

Multiplying \(-1\) above and using (4.86), we obtain the weak dynamic programming principle for \(w_2\).

**Proof of Theorem 3.1** We only need to prove for \(w_1\) and \(\bar{w}_1\), then the results of \(w_2\) and \(\bar{w}_2\) follow by a similar transformation to that used in the proof of Theorem 2.1 part (2).

a) We first show that \(w_1\) is a viscosity supersolution of (3.1) with Hamiltonian \(H_1\). Let \((t_0, x_0, \varphi) \in (0, T) \times \mathbb{R}^k \times C^{1,2}([0, T] \times \mathbb{R}^k)\) be such that \(w_1(t_0, x_0) = \varphi(t_0, x_0)\) and that \(w_1 - \varphi\) attains a strict local minimum at \((t_0, x_0)\), i.e., for some \(\delta_0 \in (0, t_0 \wedge (T-t_0)]\)

\[
(w_1 - \varphi)(t, x) > (w_1 - \varphi)(t_0, x_0), \quad \forall \ (t, x) \in O_{\delta_0}(t_0, x_0) \setminus \{(t_0, x_0)\}. \tag{4.87}
\]

We simply denote \((\varphi(t_0, x_0), D_x \varphi(t_0, x_0), D_x^2 \varphi(t_0, x_0))\) by \((y_0, z_0, \Gamma_0)\). If \(H_1(t_0, x_0, y_0, z_0, \Gamma_0) = -\infty\), then

\[
-\frac{\partial}{\partial t} \varphi(t_0, x_0) - H_1(t_0, x_0, y_0, z_0, \Gamma_0) \geq 0,
\]

holds automatically. To make a contradiction, we assume that when \(H_1(t_0, x_0, y_0, z_0, \Gamma_0) > -\infty\),

\[
\bar{\varphi} \overset{\triangle}{=} \frac{\partial}{\partial t} \varphi(t_0, x_0) + H_1(t_0, x_0, y_0, z_0, \Gamma_0) > 0. \tag{4.88}
\]

For any \((t, x, y, z, \Gamma, u, v) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^d \times \mathcal{S}_k \times \mathbb{U} \times \mathbb{V}\), one can deduce from (2.1)–(2.3) that

\[
|H(t, x, y, z, \Gamma, u, v)| \leq \frac{1}{4}|\sigma \sigma^T(t, x, u, v)|^2 + \frac{1}{4}|\Gamma|^2 + |x||b(t, x, u, v)| + \gamma \left(1 + |x|^{2p} + |y| + |y| \sigma(t, x, u, v) + |u|^{2p} + |v|^{2p}\right)
\]

\[
\leq \frac{1}{4} \gamma(1 + |x| + |u|^{2p} + |v|^{2p})^2 + \frac{1}{4}|\Gamma|^2 + \gamma + \gamma \left(1 + |x| + |u|^{2p} + |v|^{2p}\right) + \gamma \left(1 + |x|^{2p} + |y| + |u|^{2p} + |v|^{2p}\right). \tag{4.89}
\]

Set \(C_{\varphi} \overset{\triangle}{=} |y_0| + |z_0| + |\Gamma_0| = \max \{\varphi(t_0, x_0) + |D_x \varphi(t_0, x_0)| + |D_x^2 \varphi(t_0, x_0)|\}\), and fix a \(u_2 \in \partial \Omega_{\alpha}(u_0)\). For any \(u \notin \Omega_{\alpha}(u_0)\), we see from (A-u) that \(\psi(t, u) \in \mathcal{C} \times \mathcal{U}_u\), and it follows from (4.89) that

\[
\inf_{v \in \mathcal{C}_u} H(t, x_0, y_0, z_0, \Gamma_0, u, v) \leq |H(t, x_0, y_0, z_0, \Gamma_0, u, v)| = |H(t, x_0, y_0, z_0, \Gamma_0, u, \psi(t, u_2))| \leq \frac{1}{4} \left(C_{\varphi}^2 + C_{\varphi}^2 \sigma \varphi(x_0) + C \varphi(x_0) \right). \tag{4.90}
\]
Here $C(\kappa, x_0)$ denotes a generic constant, depending on $\kappa, |x_0|, T, \gamma, p$ and $|g(0)|$, whose form may vary from line to line.

Similarly, it holds for any $u \in O_\kappa(u_0)$ that

$$\inf_{v \in \mathcal{G}_u} H(t_0, x_0, y_0, z_0, \Gamma_0, u, v) \leq |H(t_0, x_0, y_0, z_0, \Gamma_0, u, v)| \leq \frac{1}{4}(C_0^2) + C_\varphi C(\kappa, \kappa_0) + C(\kappa, x_0),$$

which together with (4.96) implies that

$$H_1(t_0, x_0, y_0, z_0, \Gamma_0) \leq \sup_{u \in O_\kappa} \inf_{v \in \mathcal{G}_u} H(t_0, x_0, y_0, z_0, \Gamma_0, u, v) \leq \frac{1}{4}(C_0^2) + C_\varphi C(\kappa, \kappa_0) + C(\kappa, x_0) < \infty.$$  

Thus $\rho < \infty$.

As $\varphi \in \mathbb{C}^{1,2}([0, T] \times \mathbb{R}^k)$, we see from (4.88) that for some $\hat{u} \in \mathcal{U}$

$$\lim_{t \to t_0} \inf_{v \in \mathcal{G}_u} H(t, x, \varphi(x, t), D_x \varphi(x, t), D_x^2 \varphi(x, t), \hat{u}, v) \geq \frac{3}{4}\rho - \frac{\partial}{\partial t} \varphi(t_0, x_0).$$

Moreover, there exists a $\delta \in (0, \delta_0)$ such that

$$\inf_{v \in \mathcal{G}_u} H(t, x, \varphi(x, t), D_x \varphi(x, t), D_x^2 \varphi(x, t), \hat{u}, v) \geq \frac{1}{2}\rho - \frac{\partial}{\partial t} \varphi(t_0, x_0), \quad \forall (t, x) \in \mathcal{O}_\delta(t_0, x_0).$$

(4.91)

Let $\varphi \hat{=} \inf \{(w_1, \varphi)(t, x) : (t, x) \in \mathcal{O}_\delta(t_0, x_0) \setminus \mathcal{O}_\delta(t_0, x_0)\}$. Since the set $\mathcal{O}_\delta(t_0, x_0) \setminus \mathcal{O}_\delta(t_0, x_0)$ is compact, there exists a sequence $(t_n, x_n)_{n \in \mathbb{N}}$ on $\mathcal{O}_\delta(t_0, x_0) \setminus \mathcal{O}_\delta(t_0, x_0)$ that converges to some $(t_*, x_*) \in \mathcal{O}_\delta(t_0, x_0) \setminus \mathcal{O}_\delta(t_0, x_0)$ and satisfies

$$\rho = \lim_{n \to \infty} (w_1, \varphi)(t_n, x_n) = \varphi(t_*, x_*) \geq 0.$$  

(4.92)

Then we set $\hat{\varphi} \hat{=} \frac{\varphi \wedge \rho}{2(1 + \gamma)T} > 0$ and let $\{(t_j, x_j)\}_{j \in \mathbb{N}}$ be a sequence of $\mathcal{O}_\delta(t_0, x_0)$ such that

$$\lim_{j \to \infty} (t_j, x_j) = (t_0, x_0) \quad \text{and} \quad \lim_{j \to \infty} w_1(t_j, x_j) = w_1(t_0, x_0) = \varphi(t_0, x_0) = \lim_{j \to \infty} \varphi(t_j, x_j).$$

So one can find a $j \in \mathbb{N}$ such that

$$|w_1(t_j, x_j) - \varphi(t_j, x_j)| < \frac{5}{6} \delta.$$

(4.93)

Clearly, $\hat{\mu}_s \hat{=} \hat{u}$, $s \in [t_j, T]$ is a constant $\mathcal{U}_s$-process. Fix $\beta \in \mathfrak{B}_{t_j}$. We set

$$\tau \hat{=} (t_j, x_j, \hat{\mu}, \beta(\hat{\mu}))$$

and define

$$\tau = \tau_{\beta, \hat{\mu}} \hat{=} \inf \left\{ s \in (t_j, T) : (s, X_s) \notin \mathcal{O}_{\hat{\varphi}}(t_j, x_j) \right\} \in \mathcal{S}_{t_j, \tau}.$$  

Since $|X(T) - X(t)| \leq T - t_j \geq \frac{T - t_j}{2} > \frac{T - t_j}{2}$, the continuity of $X^\Theta$ implies that $P$-a.s.

$$\tau < T \quad \text{and} \quad (\tau \wedge s, X_{t j}^\Theta) \in \mathcal{O}_{\hat{\varphi}}(t_j, x_j) \subset \mathcal{O}_{\hat{\varphi}}(t_0, x_0), \quad \forall s \in [t_j, T];$$

(4.94)

in particular, $(\tau, X_{t j}^\Theta) \in \partial \mathcal{O}_{\hat{\varphi}}(t_j, x_j) \subset \mathcal{O}_{\hat{\varphi}}(t_0, x_0) \setminus \mathcal{O}_\varphi(t_0, x_0).$

(4.95)

The continuity of $\varphi, X^\Theta$ and (4.94) show that $Y_s \hat{=} \phi(\tau \wedge s, X_{t j}^\Theta) + \hat{\varphi}(\tau \wedge s), s \in [t_j, T]$ defines a bounded $F$-adapted continuous process. By Itô’s formula,

$$Y_s = \int_t^T f_r dr - \int_s^T \int \cdots, \quad s \in [t_j, T].$$

(4.96)
where $Z_r = \mathbf{1}_{\{r < T\}} D_x \varphi (r, X^\emptyset_t) \cdot \sigma (r, X^\emptyset_t, \tilde{u}, (\beta (\tilde{\mu}))_r)$ and

$$f_r = -\mathbf{1}_{\{r < T\}} \left\{ \tilde{\varphi} + \frac{\partial \varphi}{\partial r} (r, X^\emptyset_t) + D_x \varphi (r, X^\emptyset_t) \cdot b (r, X^\emptyset_t, \tilde{u}, (\beta (\tilde{\mu}))_r) + \frac{1}{2} \text{tr}\left( \sigma \sigma^T (r, X^\emptyset_t, \tilde{u}, (\beta (\tilde{\mu}))_r) \cdot D_x \varphi (r, X^\emptyset_t) \right) \right\}.$$ 

As $\varphi \in C^1 (\{t, T\} \times \mathbb{R}^k)$, the measurability of $b, \sigma, X^\emptyset, \tilde{u}$ and $\beta (\tilde{\mu})$ implies that both $Z$ and $f$ are $F$-progressively measurable. And one can deduce from (2.1), (2.2), (4.94) and Hölder’s inequality that

$$E \left[ \left( \int_{t_j}^T |Z_s|^2 \, ds \right)^{p/2} \right] \leq (\gamma \tilde{C}_\varphi)^p E \left[ \left( \int_{t_j}^T \left( 1 + |X^\emptyset_s| + |\tilde{u}| + [(\beta (\tilde{\mu}))_s]_v \right)^2 \, ds \right)^{p/2} \right]$$

$$\leq c_0 \tilde{C}_\varphi^p \left( \left( 1 + |x_0| + \delta + |\tilde{u}| \right)^p + \left( E \int_{t_j}^T [(\beta (\tilde{\mu}))_s]_v^2 \, ds \right)^{p/2} \right) < \infty, \quad \text{i.e. } Z \in \mathbb{D}_{1,p}^2 ([t_j, T], \mathbb{R}^d),$$

(4.97)

where $\tilde{C}_\varphi \triangleq \sup_{(t, x) \in \mathbb{D}_{1,p}^2 (t_0, x_0)} |D_x \varphi (t, x)| < \infty$. Hence, \{ $(\mathcal{Y}_s, Z_s)$ \}$_{s \in [t_j, T]}$ solves the BSDE $(t_j, \mathcal{Y}_T, f)$.

Let $\ell (x) = c_0 + c_0 |x|^{2/p}, \ x \in \mathbb{R}^k$ be the function appeared in Proposition 2.1. Let $\theta_1 : [0, T] \times \mathbb{R}^k \to [0, 1]$ be a continuous function such that $\theta_1 \equiv 0$ on $\mathcal{O}_\delta (t_0, x_0)$ and $\theta_1 \equiv 1$ on $([0, T] \times \mathbb{R}^k) \setminus \mathcal{O}_\delta (t_0, x_0)$. Also, let $\theta_2 : [0, T] \times \mathbb{R}^k \to [0, 1]$ be another continuous function such that $\theta_2 \equiv 0$ on $\mathcal{O}_\delta (t_0, x_0)$ and $\theta_2 \equiv 1$ on $([0, T] \times \mathbb{R}^k) \setminus \mathcal{O}_\delta (t_0, x_0)$. Define

$$\phi (t, x) \triangleq -\theta_1 (t, x) \ell (x) + (1 - \theta_1 (t, x)) \left( \varphi (t, x) + \varphi \theta_2 (t, x) \right), \quad \forall \ (t, x) \in [t_j, T] \times \mathbb{R}^k,$$

(4.98)

which is a continuous function satisfying $\phi \leq w_1$; given $(t, x) \in [t_j, T] \times \mathbb{R}^k$,

- if $(t, x) \in \mathcal{O}_\delta (t_0, x_0)$, (4.87) shows that $\phi (t, x) = \varphi (t, x) \leq w_1 (t, x) \leq w_1 (t, x)$;
- if $(t, x) \in \mathcal{O}_\delta (t_0, x_0)$, $\mathcal{O}_\delta (t_0, x_0)$, since $\varphi (t, x) + \varphi \theta_2 (t, x) \leq \varphi (t, x) + \varphi \leq w_1 (t, x) \leq w_1 (t, x)$ by (4.92), one can deduce from Proposition 2.1 that $\phi (t, x) \leq w_1 (t, x)$;
- if $(t, x) \notin \mathcal{O}_\delta (t_0, x_0)$, $\phi (t, x) = -\ell (x) \leq w_1 (t, x)$.

Then we can deduce from (4.95) that

$$\mathcal{Y}_T = \varphi (T, X^\emptyset_T) + \tilde{\varphi} T < \varphi (T, X^\emptyset_T) + \varphi = \phi (T, X^\emptyset_T), \quad P\text{-a.s.}$$

(4.99)

Since it holds $ds \times dP$-a.s. on $[t_j, T] \times \Omega$ that $[(\beta (\tilde{\mu}))_s]_v \leq \kappa + C_{\beta} [\bar{\mu}_s]_u = \kappa + C_{\beta} [\bar{\mu}]_u \in \mathcal{F}_u$, (4.94), (4.91) and (2.4) imply that for $ds \times dP$-a.s. $(s, \omega) \in [t_j, T] \times \Omega$

$$f_s (\omega) \leq \mathbf{1}_{\{s < \tau (\omega)\}} \left\{ -\tilde{\varphi} - \frac{1}{2} \varphi \right\} f_{t_j} (s, \omega, X^\emptyset_s (\omega), \mathcal{Y}_s (\omega), \tilde{u}_s (\omega), (\beta (\tilde{\mu}))_s (\omega))$$

$$\leq \mathbf{1}_{\{s < \tau (\omega)\}} \left\{ -\tilde{\varphi} - \frac{1}{2} \varphi \right\} f_{t_j} (s, \omega, X^\emptyset_s (\omega), \mathcal{Y}_s (\omega), \tilde{u}_s (\omega), (\beta (\tilde{\mu}))_s (\omega)) \leq f_{t_j} \left( s, \omega, \mathcal{Y}_s (\omega), \tilde{u}_s (\omega) \right),$$

(4.100)

As $f_{t_j}$ is Lipschitz continuous in $(y, z)$, Proposition 1.2 (2) implies that $P$-a.s.

$$\mathcal{Y}_s \leq \mathcal{Y}_{t_j} (\tau, \phi (T, X^\emptyset_T)), \quad \forall s \in [t_j, T].$$

Letting $s = t_j$ and using the fact that $t_j > t_0 - \frac{1}{6} T > t_0 - \frac{1}{6} \delta_0 > \frac{5}{6} \delta_0$, we obtain

$$\varphi (t_j, x_j) + \frac{5}{6} \tilde{\varphi}_0 < \varphi (t_j, x_j) + \tilde{\varphi}_j \tau \leq \mathcal{Y}_{t_j} \left( t_j, \phi (T, X^\emptyset_{t_j}, \tilde{u}_t, (\beta (\tilde{\mu}))_t) \right) \leq \text{esssup}_{\mu \in \mathcal{U}_{t_j}} \mathcal{Y}_{t_j} (\tau_j, \phi (T, X^\emptyset_{t_j}, \tilde{u}_t, (\beta (\tilde{\mu}))_t), \mu),$$

where $\tau_{\beta, \mu} \triangleq \inf \left\{ s \in (t_j, T) : (s, X^\emptyset_{t_j}, x_j, u_s, (\beta (\tilde{\mu}))_s) \notin \mathcal{O}_\delta (t_j, x_j) \right\}, \forall \mu \in \mathcal{U}_{t_j}$. Taking essential infimum over $\beta \in \mathcal{B}_{t_j}$ and applying Theorem 2.1 with $(t, x, \delta) = (t_j, x_j, \frac{2}{3} \delta)$, we see from (4.93) that

$$\varphi (t_j, x_j) + \frac{5}{6} \tilde{\varphi}_0 \leq \text{esssup}_{\beta \in \mathcal{B}_{t_j}} \mathcal{Y}_{t_j} (\tau_{\beta, \mu}, \phi (T, X^\emptyset_{t_j}, \tilde{u}_t, (\beta (\tilde{\mu}))_t), \mu) \leq w_1 (t_j, x_j) \leq \varphi (t_j, x_j) + \frac{5}{6} \tilde{\varphi}_0.$$
A contradiction appears. Therefore, \( w_1 \) is a viscosity supersolution of (3.1) with Hamiltonian \( H_1 \).

b) Next, we show that \( w_1 \) is a viscosity subsolution of (3.1) with Hamiltonian \( \overline{H}_1 \). Let \((t_0, x_0, \varphi) \in (0, T) \times \mathbb{R}^k \times C^{1,2}(\{0, T\} \times \mathbb{R}^k)\) be such that \( w_1(t_0, x_0) = \varphi(t_0, x_0) \) and that \( w_1 - \varphi \) attains a strict local maximum at \((t_0, x_0)\), i.e., for some \( \delta_0 \in (0, t_0 \wedge (T - t_0)) \)

\[
(\overline{w}_1 - \varphi)(t, x) < (\overline{w}_1 - \varphi)(t_0, x_0) = 0, \quad \forall (t, x) \in O_{\delta_0}(t_0, x_0) \setminus \{(t_0, x_0)\}.
\]

We still denote \((\varphi(t_0, x_0), D_x \varphi(t_0, x_0), D_x^2 \varphi(t_0, x_0))\) by \((y_0, z_0, \Gamma_0)\). If \( \overline{H}_1(t_0, x_0, y_0, z_0, \Gamma_0) = \infty \), then

\[
-\frac{\partial}{\partial t} \varphi(t_0, x_0) - \overline{H}_1(t_0, x_0, y_0, z_0, \Gamma_0) \leq 0,
\]

holds automatically. To make a contradiction, we assume that when \( \overline{H}_1(t_0, x_0, y_0, z_0, \Gamma_0) < \infty \),

\[
\vartheta \triangleq -\frac{\partial}{\partial t} \varphi(t_0, x_0) - \overline{H}_1(t_0, x_0, y_0, z_0, \Gamma_0) > 0.
\]

It is easy to see that

\[
\overline{H}_1(t_0, x_0, y_0, z_0, \Gamma_0) \geq \lim_{n \to \infty} \sup_{u \in \mathcal{U}} \inf_{v \in \mathcal{E}_u} H(t_0, x_0, y_0, z_0, \Gamma_0, u, v) \geq \lim_{n \to \infty} \inf_{v \in \mathcal{E}_u} H(t_0, x_0, y_0, z_0, \Gamma_0, u, v)
\]

\[
= \inf_{v \in \mathcal{O}_{\kappa}(v_0)} H(t_0, x_0, y_0, z_0, \Gamma_0, u_0, v). \tag{4.102}
\]

For any \( v \in \mathcal{O}_{\kappa}(v_0) \), one can deduce from (4.89) that

\[
|H(t_0, x_0, y_0, z_0, \Gamma_0, u, v)| \leq \frac{1}{4} (C_\varphi^2 + C_\kappa C(\kappa, x_0)) + C(\kappa, x_0),
\]

where \( C_\varphi^2 = |\varphi(t_0, x_0)| + |D_x \varphi(t_0, x_0)| + |D_x^2 \varphi(t_0, x_0)| \) as set in part (a). It then follows from (4.102) that

\[
\overline{H}_1(t_0, x_0, y_0, z_0, \Gamma_0) \geq \inf_{v \in \mathcal{O}_{\kappa}(v_0)} H(t_0, x_0, y_0, z_0, \Gamma_0, u_0, v) \geq -\frac{1}{4} (C_\varphi^2 - C_\kappa C(\kappa, x_0)) - C(\kappa, x_0) > -\infty.
\]

Thus \( \vartheta < \infty \).

Then one can find an \( m \in \mathbb{N} \) such that

\[
-\frac{\partial \varphi}{\partial t}(t_0, x_0) - \frac{7}{8} \vartheta \geq \sup_{u \in \mathcal{U}} \inf_{v \in \mathcal{E}_u} \lim_{u' \to u} \sup_{(t, x, y, z, \Gamma) \in \mathcal{O}_{1/m}(t_0, x_0, y_0, z_0, \Gamma_0)} H(t, x, y, z, \Gamma, u', v). \tag{4.103}
\]

As \( \varphi \in C^{1,2}(\{0, T\} \times \mathbb{R}^k) \), there exists a \( \delta < \frac{1}{2m} \wedge \delta_0 \) such that for any \((t, x) \in \mathcal{O}_{\delta}(t_0, x_0)\)

\[
\left| \frac{\partial \varphi}{\partial t}(t, x) - \frac{\partial \varphi}{\partial t}(t_0, x_0) \right| \leq \frac{1}{8} \vartheta \tag{4.104}
\]

and

\[
|\varphi(t, x) - \varphi(t_0, x_0)| \vee |D_x \varphi(t, x) - D_x \varphi(t_0, x)| \vee |D_x^2 \varphi(t, x) - D_x^2 \varphi(t_0, x)| \leq \frac{1}{2m},
\]

the latter of which together with (4.103) implies that

\[
-\frac{\partial \varphi}{\partial t}(t_0, x_0) - \frac{7}{8} \vartheta \geq \sup_{u \in \mathcal{U}} \inf_{v \in \mathcal{E}_u} \lim_{u' \to u} \sup_{(t, x) \in \mathcal{O}_{\delta}(t_0, x_0)} H(t, x, \varphi(t, x), D_x \varphi(t, x), D_x^2 \varphi(t, x), u', \mathcal{P}_u(u)),
\]

Then for any \( u \in \mathcal{U} \), there exists a \( \mathcal{P}_u(u) \in \mathcal{E}_u^{m} \) such that

\[
-\frac{\partial \varphi}{\partial t}(t_0, x_0) - \frac{3}{4} \vartheta \geq \sup_{(t, x) \in \mathcal{O}_\lambda(t_0, x_0)} H(t, x, \varphi(t, x), D_x \varphi(t, x), D_x^2 \varphi(t, x), u', \mathcal{P}_u(u), u'),
\]

and we can find a \( \lambda(u) \in (0, 1) \) such that for any \( u' \in O_{\lambda(u)}(u) \)

\[
-\frac{\partial \varphi}{\partial t}(t_0, x_0) - \frac{5}{8} \vartheta \geq \sup_{(t, x) \in \mathcal{O}_\lambda(t_0, x_0)} H(t, x, \varphi(t, x), D_x \varphi(t, x), D_x^2 \varphi(t, x), u', \mathcal{P}_u(u)). \tag{4.105}
\]
Set $\bar{\lambda}(u_0) = \lambda(u_0)$ and $\bar{\lambda}(u) = \lambda(u) \wedge (\frac{1}{2}[u]_U)$ for any $u \in U \setminus \{u_0\}$. Since the separable metric space $U$ is Lindelöf, \( \{D(u) \triangleq O_{\bar{\lambda}(u)}(u)\}_{u \in U} \) has a countable subcollection \( \{D(u_i)\}_{i \in \mathbb{N}} \) to cover $U$. It is clear that

$$
P(u) \triangleq \sum_{i \in \mathbb{N}} 1_{\{u \in D(u_i) \setminus \subseteq_{i} D(u_j)\}} \mathbb{P}_{\beta}(u_i) \in \mathbb{V}, \quad \forall u \in U$$

defines a $\mathcal{B}(U)/\mathcal{B}(V)$-measurable function.

Given $u \in U$, there exists an $i \in \mathbb{N}$ such that $u \in D(u_i) \setminus \bigcup_{j \neq i} D(u_j)$. If $u_i = u_0$,

$$[\mathbb{P}(u)]_V = [\mathbb{P}_0(u_i)]_V \leq k + m[u_i]_U = k \leq k + m[u]_U. \quad \text{(4.106)}$$

On the other hand, if $u_i \neq u_0$, then $[u_i]_U \leq [u]_U + \rho(u_i, u) \leq [u]_U + \bar{\lambda}(u_i) \leq [u]_U + \frac{1}{2}[u]_U$, and it follows that

$$[\mathbb{P}(u)]_V = [\mathbb{P}_0(u_i)]_V \leq k + m[u_i]_U \leq k + 2m[u]_U. \quad \text{(4.107)}$$

Also, we see from (4.105) that

$$-\frac{\partial \varphi}{\partial t}(t_0, x_0) - \frac{5}{8} \varphi \geq \sup_{(t, x) \in \overline{O}_\delta(t_0, x_0)} H(t, x, \varphi(t, x), D_x \varphi(t, x), D_x^2 \varphi(t, x), u, \mathbb{P}_0(u_i))$$

$$= \sup_{(t, x) \in \overline{O}_\delta(t_0, x_0)} H(t, x, \varphi(t, x), D_x \varphi(t, x), D_x^2 \varphi(t, x), u, \mathbb{P}(u)),$$

which together with (4.104) implies that

$$-\frac{\partial \varphi}{\partial t}(t, x) - \frac{1}{2} \varphi \geq H(t, x, \varphi(t, x), D_x \varphi(t, x), D_x^2 \varphi(t, x), u, \mathbb{P}(u)), \quad \forall (t, x) \in \overline{O}_\delta(t_0, x_0), \quad \forall u \in U. \quad \text{(4.108)}$$

Similar to (4.92), we set $\varphi \triangleq \min \{((\varphi - \overline{\varphi})(t, x) : (t, x) \in \overline{O}_\delta(t_0, x_0) \setminus \overline{O}_{\frac{\delta}{2}}(t_0, x_0)\} > 0$ and $\varphi \triangleq \frac{\varphi \wedge \varphi}{2(1 + \gamma)^T}$ > 0. Let $\{t_j, x_j\}_{j \in \mathbb{N}}$ be a sequence of $O_{\frac{\delta}{2}}(t_0, x_0)$ such that

$$\lim_{j \to \infty} (t_j, x_j) = (t_0, x_0) \quad \text{and} \quad \lim_{j \to \infty} w_1(t_j, x_j) = \overline{\varphi}_1(t_0, x_0) = \varphi(t_0, x_0) = \lim_{j \to \infty} \varphi(t_j, x_j).$$

So one can find a $j \in \mathbb{N}$ that

$$|w_1(t_j, x_j) - \varphi(t_j, x_j)| < \frac{5}{6} \varphi(t_0). \quad \text{(4.109)}$$

For any $\mu \in U_{t_j}$, the measurability of function $\mathbb{P}$ shows that $\overline{\beta}(\mu) \triangleq \mathbb{P}(\mu_s), s \in [t_j, T]$ is a $\mathbb{V}$-valued, $F$-progressively measurable process. By (4.106) and (4.107),

$$[\overline{\beta}(\mu)]_s = [\mathbb{P}(\mu_s)]_s \leq \kappa + 2m[\mu_s]_U, \quad \forall s \in [t_j, T].$$

Let $E \int_{t_j}^{T} [\mu_s]_U^q ds < \infty$ for some $q > 2$. It then follows that

$$E \int_{t_j}^{T} [\overline{\beta}(\mu)]_s^q ds \leq 2^{q-1}\kappa^qT + 2^{2q-1}m^qE \int_{t_j}^{T} [\mu_s]_U^q ds < \infty.$$

So $\overline{\beta}(\mu) \in \mathbb{V}_{t_j}$. Let $\mu^1, \mu^2 \in U_{t_j}$ such that $\mu^1 = \mu^2, ds \times dP$-a.s. on $[[t_j, \tau][\tau, T]]_A$ for some $\tau \in S_{t_j, T}$ and $A \in F_\tau$. Then it directly follows that $\overline{\beta}(\mu^1)_s = \mathbb{P}(\mu^1)_s = \overline{\beta}(\mu^2)_s, ds \times dP$-a.s. on $[[t_j, \tau][\tau, T]]_A$. Hence, $\overline{\beta} \in \mathcal{B}_{t_j}$.

Let $\mu \in U_{t_j}$. We set $\Theta_{1, \mu} \triangleq (t_j, x_j, \mu, \overline{\beta}(\mu))$ and define

$$\tau_{\mu} = \tau_{1, \mu} \triangleq \inf \{s \in (t_j, T) : (s, X_{s}^{\Theta}) \notin O_{\frac{\delta}{4}}(t_j, x_j)\} \in S_{t_j, T}.
As \(|(T, X^\tau^\mu) - (t_j, x_j)| \geq T - t_j \geq T - t_0 - |t_j - t_0| > \delta_0 - \frac{\delta}{6} > \frac{5}{6} \delta|, the continuity of \(X^\tau^\mu\) implies that \(P\)-a.s.

\[
\tau^\mu \leq \frac{5}{6} t_0 \geq \frac{5}{6} \delta t_0 > \frac{5}{6} \delta t_0 > \frac{5}{6} \delta t_0.
\]

Letting \(s = t_j\) and using the fact that \(t_j > t_0 - \frac{1}{6} \delta > t_0 - \frac{1}{6} \delta > \frac{5}{6} \delta t_0, we obtain

\[
\varphi(t_j, x_j) - \frac{5}{6} \delta t_0 \geq \varphi(t_j, x_j) - \delta t_0 = Y^t_j \geq Y^t_j, x_j, \mu, \beta(\mu)\left(\tau^\mu, \phi(\tau^\mu, X^\tau^\mu, s, \beta(\mu))\right), \quad P\)-a.s.
\]

Taking essential supremum over \(\mu \in \mathcal{U}_{t_j}\) and applying Theorem 2.1 with \((t, x, \delta) = (t_j, x_j, \frac{5}{6} \delta), we see from (4.109) that \(P\)-a.s.

\[
\varphi(t_j, x_j) - \frac{5}{6} \delta t_0 \geq \sup_{\mu \in \mathcal{U}_{t_j}} Y^t_j, x_j, X^\tau^\mu, s, \beta(\mu)\left(\tau^\mu, \phi(\tau^\mu, X^\tau^\mu, s, \beta(\mu))\right) \geq \inf_{\beta \in \beta^\mu} \sup_{\mu \in \mathcal{U}_{t_j}} Y^t_j, x_j, X^\tau^\mu, s, \beta(\mu)\left(\tau^\mu, \phi(\tau^\mu, X^\tau^\mu, s, \beta(\mu))\right)
\]

\[
\geq w_1(t_j, x_j) > \varphi(t_j, x_j) - \frac{5}{6} \delta t_0,
\]

where \(w_1(t_j, x_j) \geq \inf \left\{ s \in (t_j, T): (s, X^t_j, x_j, \mu, \beta(\mu) \notin O_{\frac{5}{6} \delta}(t_j, x_j) \right\}. A contradiction appears. Therefore, \(w_1\) is a viscosity supersolution of (3.1) with Hamiltonian \(H_1\).
References


