Stability and attraction of critical points

Math 3950 *Nonlinear Dynamical Systems* - 2/2/05

Consider a dynamical system $\dot{x} = f(x)$ where f is locally Lipschitz on an open set $E \subset \mathbb{R}^n$. Assume that $f(p) = 0$; that is, p is a critical point of the system.

Definition: The point p is stable if for all $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that if $x_0 \in B_\delta(p)$, then $\Phi(t, x_0) \in B_\epsilon(p)$ for all $t \geq 0$.

Remark: Note that by construction, $\delta \leq \epsilon$.

Definition: The point p is asymptotically stable if it is stable and, further, there exists μ such that for all $x_0 \in B_\mu(p), \Phi(t, x_0) \to p$ as $t \to \infty$.

Now, Perko gives definitions of an *attracting set* and an *attractor* in Section 3.2. If we apply these definitions specifically to the critical point p, then we obtain the following.

Definition: The point p is both an attracting set and an attractor if there exists an open set U such that $p \in U$ and such that for all $x_0 \in U$, $\Phi(t, x_0) \in U$ for all $t \geq 0$ and $\Phi(t, x_0) \to p$ as $t \to \infty$.

Remark: Note that by definition, a critical point is either an attracting set and an attractor or neither.

Recall the example

$$\begin{aligned} \dot{r} &= r(1-r) \\ \dot{\theta} &= \sin^2(\theta/2). \end{aligned} \tag{1}$$

If we examine the above definitions, it is apparent that the critical point $(1, 0)$ is not stable (and therefore is unstable, and is not asymptotically stable), since (as discussed in class) the flow on $\{r = 1\}$ sends points that start arbitrarily close to $(1, 0)$ all the way around $\{r = 1\}$ before they converge back to $(1, 0)$. However, *contrary to the discussion in class*, $(1, 0)$ **is** an attractor. To define the open set U necessary to satisfy the definition of attractor, consider an initial point (r_0, θ_0) with $r > 1$ but close to 1 and $\theta < 0$ but close to 0; i.e., the point is just below the positive x-axis, just outside of $\{r = 1\}$. Follow the trajectory from this point *backwards in time*, under the time-reversed flow of system (1), until it flows around to small positive θ (which will occur if (r_0, θ_0) is chosen appropriately). Use the resulting trajectory as part of the boundary of U. To form the rest of the boundary of U, connect the starting and ending points of this trajectory to the positive x-axis, $\{\theta = 0\}$, with vertical segments. U will be positively invariant, since $\dot{r} < 0$ for $r > 1$ and since trajectories cannot cross by uniqueness. Finally, to avoid the fact that $\Phi(t, (0,0))$ does NOT converge to $(1,0)$ as $t \to \infty$, we exclude a small (circular) ball, call it B_0, around $(0,0)$ from U. As long as we omit all of the boundary of U from U, U will be open, and this exclusion leaves U positively invariant, since $\dot{r} > 0$ on the boundary of B_0. (Feel free to ask me for further clarification.)

The above example proves that an attracting critical point need not be stable (nor asymptotically stable, of course). While I do not have a counterexample in hand, I would treat attracting and asymptotically stable as distinct notions entirely for a critical point, based on Perko's definition of attracting. That is, Perko's definition of attracting requires that U is positively invariant, while the definition of asymptotically stable does not require the existence of a positively invariant set. Other definitions of attracting introduce relationships between these notions. In particular, if one drops the invariance from the definition of an attracting critical point, then one can define an asymptotically stable critical point as one that is stable and attracting (as in Strogatz's book). Alternatively, a set can be defined as attracting if it is the ω-limit set of a *set* that contains it, in which case being attracting is equivalent to being asymptotically stable for a critical point.

All of this suggests that classifying a critical point as stable or asymptotically stable is a more universal means of description than worrying about whether it is attracting or not. Attracting is a more important notion for larger sets, where it takes the place of stability/asymptotic stability.