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A cubic system with thirteen limit cycles [☆]

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ABSTRACT

We construct a planar cubic system and demonstrate that it has at least 13 limit cycles. The construction is essentially based on counting the number of zeros of some Abelian integrals.

1. Introduction

The second part of Hilbert's 16th problem is to ask the exact upper bound of the number of limit cycles, denoted as $H(n)$, and to determine their relative positions for all the polynomial vector fields of degree n in the plane. In the past decades, extensive investigation on this topic has been carried out, various kinds of particular systems have been discussed and many research articles have been published. However, up to now, this problem remains to be so difficult and elusive that we even do not know if $H(2)$ is finite or not.

While it is very open to obtain $H(n)$ even for $n = 2$, investigation has been gradually turned to look for the lower bound of $H(n)$. For a given number n, attempts to update the lower bounds of $H(n)$ have become one of the most interesting topics of research. Some known results are as follows: it is shown in [1,8] that $H(2) \geqslant 4$ and $H(3) \geqslant 11$ in [7]. In [3], it is proved that $H(n)$ grows at least as rapidly as $n^2 \log n$. Recently, by calculating the focus values of some cubic systems, in [9], it is demonstrated that $H(3) \geqslant 12$. For the latest development about $H(n)$, we refer the reader to, say, [2,6,9], etc. In fact, one can obtain a detailed list of reference with great ease via e-resources.

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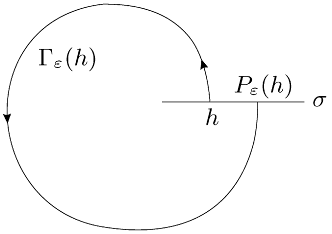

Fig. 1. Construction of the displacement function.

In this paper, we will explicitly construct a cubic system and theoretically prove that 13 limit cycles can be bifurcated under small perturbation. More precisely, consider the following perturbed planar Hamiltonian system having the form

$$\frac{dx}{dt} = -\frac{\partial H(x,y)}{\partial y} + \varepsilon P(x,y), \qquad \frac{dy}{dt} = \frac{\partial H(x,y)}{\partial x} + \varepsilon Q(x,y), \tag{1}$$

where $H(x,y)$, $P(x,y)$ and $Q(x,y)$ are polynomials in (x,y) and ε is a small parameter. In this paper, we prove the following

Main Theorem. *There exists a cubic system of the form* (1), *where $H(x,y)$ is a polynomial of degree* 4, *and $P(x,y)$ and $Q(x,y)$ are polynomials of degree* 3, *such that it has at least* 13 *limit cycles in the plane for sufficiently small parameter ε.*

2. The sketch of the construction of the system

Given a system of the form (1), we assume that the level curves $\Gamma(h) := \{(x,y) \mid H(x,y) = h\}$, which continuously depend on the parameter $h \in (a,b)$, have at least one family of closed orbits. Then the Abelian integral $I(h)$ of system (1) given by

$$I(h) = \oint_{\Gamma(h)} P(x,y)\,dy - Q(x,y)\,dx \tag{2}$$

is well defined in (a,b).

Take a segment σ, which is transversal to each oval $\Gamma(h)$, and parameterize σ by the values of $H(x,y)$. Denote by $\Gamma_\varepsilon(h)$ a piece of the orbit of the system (1) between the starting point h on σ and the next intersection point $P_\varepsilon(h)$ with σ (see Fig. 1). As usual, we call $d_\varepsilon(h) = P_\varepsilon(h) - h$ the displacement function of (1). Obviously, if $d_\varepsilon(h) \not\equiv 0$ and $d_\varepsilon(h_0) = 0$, then the curve $\Gamma_\varepsilon(h_0)$ is a limit cycle of the system (1).

In what follows, we first recall the Poincaré–Pontryagin Theorem and then we present a sketch of the construction of the system satisfying our requirements.

Theorem 1 *(Poincaré–Pontryagin). The displacement function $d_\varepsilon(h)$ has the following asymptomatic expansion*:

$$d_\varepsilon(h) = \varepsilon\big(I(h) + \varepsilon\varphi_\varepsilon(h)\big), \quad \varepsilon \to 0,$$

where $\varphi_\varepsilon(h)$ is analytic and uniformly bounded for (h,ε) in a compact neighborhood of $(h,0)$.

As one of the most classical theorems in qualitative theory of ordinary differential equations, the Poincaré–Pontryagin Theorem can be found in many standard textbooks. The theorem has the following immediate corollaries:

(1) If h_0 is one simple zero of $I(h)$, then for sufficiently small ε, $d_\varepsilon(h)$ also has one simple zero close to h_0. Therefore system (1) has one limit cycle close to $\Gamma(h_0)$;
(2) If $I(h)$ is well defined in (h_1, h_2) and $I(h_1^+)I(h_2^-) < 0$, then for sufficiently small ε, $d_\varepsilon(h)$ has at least one zero. It follows that system (1) has at least one limit cycle.

To construct a cubic system having at least 13 limit cycles, in (1) we take the Hamiltonian function

$$\begin{aligned} H(x,y) &= \int_0^x x(x+1)(x-\lambda)\,dx + \frac{y^4}{4} - \frac{k^2y^2}{2} \\ &= F(x) + \left(\frac{y^2-k^2}{2}\right)^2 - \frac{k^4}{4} \end{aligned} \tag{3}$$

and the perturbation functions

$$P(x,y) = 0, \qquad Q(x,y) = y\big(\alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 y^2\big), \tag{4}$$

where λ, k, α_1, α_2, α_3 and α_4 are all real numbers, and

$$F(x) = \int_0^x x(x+1)(x-\lambda)\,dx = \frac{x^4}{4} + \frac{1-\lambda}{3}x^3 - \frac{\lambda}{2}x^2.$$

Then the Abelian integral under consideration takes the form:

$$I(h) = \int_{\Gamma(h)} y\big(\alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 y^2\big)\,dx = \int_{\Gamma(h)} \omega. \tag{5}$$

Notice that system (1) with $\varepsilon = 0$ has nine singular points P_i $(i = 1, \ldots, 9)$ which are identified as follows:

$$\begin{aligned} &P_1 = (\lambda, k), && P_2 = (0, k), && P_3 = (-1, k), \\ &P_4 = (\lambda, 0), && P_5 = (0, 0), && P_6 = (-1, 0), \\ &P_7 = (\lambda, -k), && P_8 = (0, -k), && P_9 = (-1, -k). \end{aligned}$$

Correspondingly, we can obtain the values of the Hamiltonian at these points. Namely,

$$\begin{gathered} H(P_1) = H(P_7) = -\frac{1}{12}\lambda^3(\lambda+2) - \frac{1}{4}k^4, \\ H(P_2) = H(P_8) = -\frac{1}{4}k^4, \\ H(P_3) = H(P_9) = -\frac{1}{12}(2\lambda+1) - \frac{1}{4}k^4, \end{gathered}$$

and

$$H(P_4) = -\frac{1}{12}\lambda^3(\lambda+2), \qquad H(P_5) = 0, \qquad H(P_6) = -\frac{1}{12}(2\lambda+1).$$

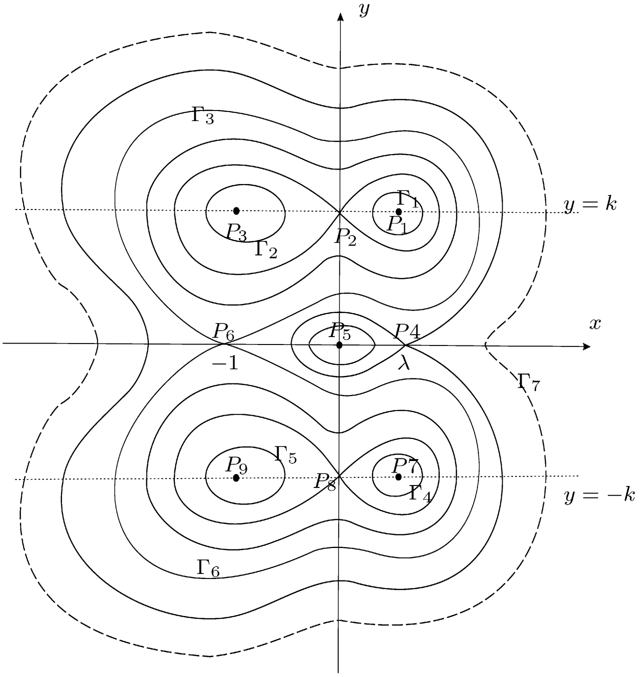

Fig. 2. The phase portraits of the unperturbed system.

If we choose $0 < \lambda < 1$ and $k > 10$, then we have

$$H(P_3) < H(P_1) < H(P_2) < H(P_6) < H(P_4) < 0.$$

The portrait of the system (1) with $\varepsilon = 0$ is described in Fig. 2.

The unperturbed system (1), i.e. $\varepsilon = 0$ in (1), is symmetric with respect to x axis (and the perturbation is symmetric with respect to x axis, too), and its portrait consists of at least 7 period annuli.

(i) If $H(P_1) < h < H(P_2)$, then there is a family of closed orbits surrounding P_1 and we have one period annulus $\{\Gamma_1(h)\}$. Symmetrically, there is a family of closed orbits surrounding P_7 and we have one period annulus $\{\Gamma_4(h)\}$.

(ii) If $H(P_3) < h < H(P_2)$, then there is a family of closed orbits surrounding P_3 and we have one period annulus $\{\Gamma_2(h)\}$. Symmetrically, there is a family of closed orbits surrounding P_9 and we have one period annulus $\{\Gamma_5(h)\}$.

(iii) If $H(P_2) < h < H(P_6)$, then there is a family of closed orbits surrounding P_1, P_2 and P_3, and we have one period annulus $\{\Gamma_3(h)\}$. Symmetrically, there is a family of closed orbits surrounding P_7, P_8 and P_9, and we have one period annulus $\{\Gamma_6(h)\}$.

(iv) If $H(P_4) < h < +\infty$, then there is a family of closed orbits surrounding all these nine singular points, and we have one period annulus $\{\Gamma_7(h)\}$.

Corresponding to $\Gamma_j(h)$, defined above for $1 \leqslant j \leqslant 7$, we denote the Abelian integrals $I_j(h)$ as follows:

$$I_j(h) = \int\limits_{\Gamma_j(h)} y\left(\alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 y^2\right) dx, \quad \text{for } 1 \leqslant j \leqslant 7. \tag{6}$$

Now by the Poincaré–Pontryagin Theorem, to prove our main result, it suffices to show the following statements.

Proposition 1. *There exist suitable k, λ, α_1, α_2, α_3 and α_4 such that the total number of zeros of $I_j(h)$, $j = 1, \ldots, 7$, is at least 13. More exactly, for $k > 10$, $0 < \lambda < 1$, the following statements hold.*

- (i) *If $h \in (H(P_3), H(P_2))$, then $I_2(h)$ has at least 1 simple zero. Symmetrically, $I_5(h)$ also has at least 1 simple zero in this interval.*
- (ii) *If $h \in (H(P_2), H(P_6))$, then $I_3(h)$ has at least 5 simple zeros. Symmetrically, $I_6(h)$ also has at least 5 simple zeros in this interval.*
- (iii) *$I_7(H(P_4))$ has different sign with $I_7(+\infty)$. Thus, $I_7(h)$ has at least 1 zero in the interval $(H(P_4), +\infty)$.*

The proof of Proposition 1 will be given in the next sections.

3. The zeros of $I_i(h)$, for $i = 2, 3, 5, 6$

In this section we will study the numbers of zeros of $I_2(h)$ and $I_3(h)$. Due to the symmetry of the system when $\varepsilon = 0$, $I_5(h)$ and $I_6(h)$ have the same numbers of zeros as $I_2(h)$ and $I_3(h)$ have, respectively. The integral $I_7(h)$ will be considered in the next section.

We first consider the following Hamiltonian system

$$\begin{cases} \dot{x} = -2\tilde{y}, \\ \dot{\tilde{y}} = x(x+1)(x-\lambda), \end{cases} \tag{7}$$

where $0 < \lambda < 1$. This system has been discussed in [5]. Notice that the Hamiltonian function takes the form

$$\tilde{H}(x, \tilde{y}) = \tilde{y}^2 + \int_0^x x(x+1)(x-\lambda)\,dx = \tilde{y}^2 + F(x)$$

and the system has three singular points and three period annuli in the whole plane specified as follows:

$\tilde{\Gamma}_1(h)$: when $-\frac{1}{12}\lambda^3(\lambda+2) < h < 0$, there is a family of closed orbits surrounding $(\lambda, 0)$.
$\tilde{\Gamma}_2(h)$: when $-\frac{1}{12}(2\lambda+1) < h < 0$, there is a family of closed orbits surrounding $(-1, 0)$.
$\tilde{\Gamma}_3(h)$: when $0 < h < +\infty$, there is a family of closed orbits surrounding $(\lambda, 0)$, $(0, 0)$ and $(-1, 0)$.

Denote by

$$\tilde{I}_j(h) = \int_{\tilde{\Gamma}_j(h)} \tilde{y}\big(\beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 \tilde{y}^2\big)\,dx, \quad \text{for } 1 \leqslant j \leqslant 3, \tag{8}$$

where β_1, β_2, β_3 and β_4 are real numbers. We assume that the orientation of the Abelian integrals is clockwise.

Concerning the number of zeros of the Abelian integrals $\tilde{I}_j(h)$ given by (8), we have the following conclusion.

Lemma 1. *There exist real numbers λ, β_1, β_2, β_3 and β_4 such that $\tilde{I}_2(h)$ has at least 1 simple zero in $(-\frac{1}{12}(2\lambda+1), 0)$ and $\tilde{I}_3(h)$ has at least 5 simple zeros in $(0, \infty)$.*

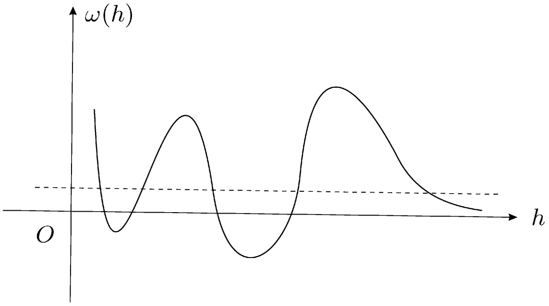

Fig. 3. The behavior of $\tilde{I}_3(h)$.

Proof. According to Theorem 1.1 of [5], when $0 < \lambda - \lambda^* \ll 1$, where $\lambda^* \sim 0.0549$ is the root of the formula (10) in [5], there exist real numbers β_1, β_2 and β_3 such that

$$\int\limits_{\tilde{\Gamma}_2(h)} \tilde{y}\big(\beta_1 + \beta_2 x + \beta_3 x^2\big)\,dx = 0 \tag{9}$$

and

$$\int\limits_{\tilde{\Gamma}_3(h)} \tilde{y}\big(\beta_1 + \beta_2 x + \beta_3 x^2\big)\,dx = 0, \tag{10}$$

respectively, have 1 simple root in $(-\frac{1}{12}(2\lambda + 1), 0)$ and 4 simple roots in $(0, +\infty)$. Therefore, it is obvious that for sufficiently small β_4, the equation $\tilde{I}_2(h) = 0$, as the perturbation equation of (9), has 1 simple zero.

To study the zeros of $\tilde{I}_3(h)$, first of all, we have the following estimations. When $h \to +\infty$,

$$\int\limits_{\tilde{\Gamma}_3(h)} \tilde{y}\,dx \sim \lambda_1 h^{3/4}, \qquad \int\limits_{\tilde{\Gamma}_3(h)} x\tilde{y}\,dx \sim \frac{\lambda_1(\lambda - 1)}{3} h^{3/4},$$

$$\int\limits_{\tilde{\Gamma}_3(h)} x^2\tilde{y}\,dx \sim \lambda_3 h^{5/4}, \qquad \int\limits_{\tilde{\Gamma}_3(h)} \tilde{y}^3\,dx \sim \lambda_4 h^{7/4},$$

where λ_1, λ_3 and λ_4 are positive constants. The proof of these estimations is extremely similar to the proof of Lemmas 2.6 and 2.8 in [4] and we omit the proof here.

From these estimations we see that, when $h \to \infty$,

$$\frac{\int_{\tilde{\Gamma}_3(h)} \tilde{y}(\beta_1 + \beta_2 x + \beta_3 x^2)\,dx}{\int_{\tilde{\Gamma}_3(h)} \tilde{y}^3\,dx} \to 0.$$

Without loss of the generality, suppose that $\int_{\tilde{\Gamma}_3(h)} \tilde{y}(\beta_1 + \beta_2 x + \beta_3 x^2)\,dx$ is positive when h is sufficiently large, then for any $\beta_4 < 0$ and $|\beta_4| \ll 1$, the equation

$$\omega(h) = \frac{\int_{\tilde{\Gamma}_3(h)} \tilde{y}(\beta_1 + \beta_2 x + \beta_3 x^2)\,dx}{\int_{\tilde{\Gamma}_3(h)} \tilde{y}^3\,dx} = -\beta_4$$

has 5 simple roots. Namely, the equation $\tilde{I}_3(h) = 0$ has at least 5 simple roots. See Fig. 3. □

With the above discussion about the number of zeros of $\tilde{I}_j(h)$, we can study the Abelian integrals $I_j(h)$ of the original system (1). To this end, we assume that λ, β_1, β_2, β_3 and β_4 are real constants which satisfy the conditions of Lemma 1. Choose α_j as follows

$$\alpha_1 = \frac{3\beta_4}{2k} + \frac{\beta_1}{k^5}, \qquad \alpha_2 = \frac{\beta_2}{k^5}, \qquad \alpha_3 = \frac{\beta_3}{k^5}, \qquad \alpha_4 = -\frac{\beta_4}{2k^3}. \tag{11}$$

We have the following lemma.

Lemma 2. *Let α_i be given as in* (11) *and $I_i(h)$ in* (6)*. Then for sufficiently large number k, $I_2(h)$ has at least* 1 *simple zero and $I_3(h)$ at least* 5 *simple zeros.*

Proof. For $y > 0$, let

$$y^2 - k^2 = 2\tilde{y}. \tag{12}$$

Then the Hamiltonian function of (1) can be rewritten as

$$H(x,y) = F(x) + \tilde{y}^2 - \frac{1}{4}k^4 = \tilde{H}(x,\tilde{y}) - \frac{1}{4}k^4.$$

Suppose that the largest root of the equation $\tilde{I}_3(h) = 0$ is h^*. We choose a constant k_1 such that

$$h^* + 1 - \frac{1}{4}k_1^4 < H(P_6).$$

Denote by $\tilde{h} = h + \frac{1}{4}k^4$. If $k > k_1$, then for any h satisfying $\tilde{h} \in (0, h^*+1)$ (resp. $\tilde{h} \in (-\frac{1}{12}(2\lambda+1), 0)$), the curve $\{(x,y) \mid H(x,y) = h\}$ is precisely $\Gamma_3(h)$ (resp. $\Gamma_2(h)$) of the unperturbed system of (1).

Now we take

$$k_2 = 2\left(h^* + 1 + \frac{1}{12}(2\lambda+1)\right)^{1/4}.$$

Then for any h satisfying $\tilde{h} \in (-\frac{1}{12}(2\lambda+1), 0) \cup (0, h^*+1)$, the point $(x,\tilde{y})$ on $\tilde{\Gamma}_3(\tilde{h})$ satisfies the following relation

$$4\tilde{y}^2 = 4\big(\tilde{h} - F(x)\big) < 4\left(h^* + 1 + \frac{1}{12}(2\lambda+1)\right) < k_2^4.$$

Equivalently, $2\tilde{y} < k_2^2$. Therefore, by taking k big enough, $k > k^* > k_2$, we can have the following Taylor expansions with the Lagrange remainders:

$$\begin{aligned}
\big(2\tilde{y} + k^2\big)^{\frac{1}{2}} &= k\left(1 + \frac{2\tilde{y}}{k^2}\right)^{\frac{1}{2}} \\
&= k\left(1 + \frac{\tilde{y}}{k^2} + \sum_{n=2}^{+\infty} \frac{(-1)^{n-1}(2n-3)!!}{n!} \frac{\tilde{y}^n}{k^{2n}}\right), \\
\big(2\tilde{y} + k^2\big)^{\frac{3}{2}} &= k^3\left(1 + \frac{2\tilde{y}}{k^2}\right)^{\frac{3}{2}} \\
&= k^3\left(1 + \frac{3\tilde{y}}{k^2} + \frac{3\tilde{y}^2}{2k^4} + 3\sum_{n=3}^{+\infty} \frac{(-1)^n(2n-5)!!}{n!} \frac{\tilde{y}^n}{k^{2n}}\right).
\end{aligned} \tag{13}$$

Notice that $\int_{\Gamma_j(\tilde h)} x^i\tilde y^{2l}\,dx=0$, for $j=2,3$, $i,l\in\{0,1,2,\ldots\}$. We substitute the relations (13) into the expression

$$I_j\left(h+\frac{k^4}{4}\right)=\int\limits_{\Gamma_j(h+\frac{k^4}{4})} y\left(\alpha_1+\alpha_2x+\alpha_3x^2+\alpha_4y^2\right)dx.$$

With some straightforward calculation, we can express it as follows:

$$I_j\left(h+\frac{k^4}{4}\right)=\frac{1}{k^6}\tilde I_j(\tilde h)+\frac{1}{k^{10}}\Psi_j\left(\tilde h,\frac{1}{k}\right), \tag{14}$$

where

$$\tilde I_j(\tilde h)=\int\limits_{\tilde\Gamma_j(\tilde h)}\left((\beta_1+\beta_2x+\beta_3x^2)\tilde y+\beta_4\tilde y^3\right)dx,$$

$$\begin{aligned}\Psi_j\left(\tilde h,\frac{1}{k}\right)&=3\beta_4\sum_{m=0}^{+\infty}\frac{(4m+5)!!}{(2m+4)!}\frac{1}{k^{4m}}\int\limits_{\tilde\Gamma_j(\tilde h)}\tilde y^{2m+5}\,dx\\&\quad+\sum_{m=0}^{+\infty}\frac{(4m+3)!!}{(2m+3)!}\frac{1}{k^{4m}}\int\limits_{\tilde\Gamma_j(\tilde h)}\left(\beta_1+\beta_2x+\beta_3x^2\right)\tilde y^{2m+3}\,dx.\end{aligned}\tag{15}$$

Let us show that there exists a constant $\tilde k$, such that when $k>\tilde k$ the function $\Psi_2(\tilde h,\frac{1}{k})$ is bounded if $\tilde h$ is restricted to a compact region in $(-\frac{1}{12}(2\lambda+1),0)$ and $\Psi_3(\tilde h,\frac{1}{k})$ is bounded when $\tilde h$ is restricted to a compact region in $(0,h^*+1)$.

To see this point, observe that in these regions, x, $\tilde y$ and $\tilde\Gamma_j(\tilde h)$ are bounded. Therefore there exist constants c and M such that the integrals in the above summation are bounded by cM^{2m}. Notice that $\frac{(4m+5)!!}{(2m+4)!}\leqslant 2^{2m}$ and $\frac{(4m+3)!!}{(2m+3)!}\leqslant 2^{2m}$. It follows that these two series in the expression of Ψ_j are controlled by a constant series $\sum_{m=0}^{+\infty}(\frac{2M}{k^2})^{2m}$. Consequently, there exists a constant $\tilde k$ such that when $k>\tilde k$ the two series in (15) converge uniformly with respect to $\tilde h$, where $\tilde h$ belongs to the corresponding regions, and the functions $\Psi_j(\tilde h,\frac{1}{k})$ are bounded.

Now from the expression of $I_j(h)$, we see that there exists a constant k_3 such that if $k>k_3$, then $I_2(h)$ has one simple zero in $(-\frac{1}{12}(2\lambda+1)-\frac14k^4,-\frac14k^4)$ and $I_3(h)$ has 5 simple zeros in $(-\frac14k^4,h^*+1-\frac14k^4)$. $\square$

Since the original system as well as the ω is symmetric with respect to x axis, it follows that $I_5(h)$ has 1 zero and $I_6(h)$ has 5 simple zeros. In other words, altogether $I_j(h)$, for $j=1,\ldots,6$, have at least 12 zeros.

4. The zeros of $I_7(h)$

Now we study the zeros of the Abelian integral $I_7(h)$ defined in (6). We will prove that $I_7(\infty)\cdot I_7(H(P_4))<0$, which implies that $I_7(h)$ has at least 1 zero when $h\geqslant H(P_4)$.

We have the following result.

Lemma 3. *Let $I_7(h)$ be given in (6) where α_i are specified in (11). Then there exists a constant k_4 such that for any $k>k_4$, when h is sufficiently large, $I_7(h)$ has the different sign with β_4.*

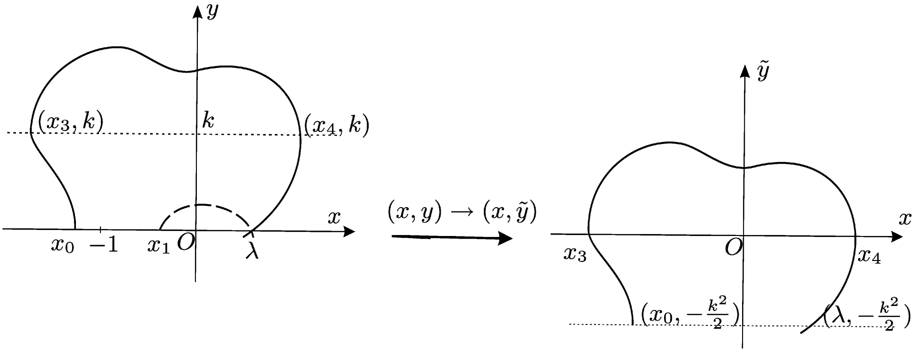

Fig. 4. The curve $\{(x,y)\mid H(x,y)=H(P_4)\}$.

Proof. Very similar to Lemma 2.6 in [4], if $h\to+\infty$, we have

$$\int\limits_{\Gamma_7(h)} y\,dx\sim\lambda_5h^{1/2},\qquad \int\limits_{\Gamma_7(h)} xy\,dx\sim\lambda_6h^{3/4},$$

$$\int\limits_{\Gamma_7(h)} x^2y\,dx\sim\lambda_7h,\qquad \int\limits_{\Gamma_7(h)} y^3\,dx\sim\lambda_8h,$$

where λ_5, λ_6, λ_7 and λ_8 are positive constants. Thus if $h\to\infty$, then

$$I_7(h)\sim\left(\frac{\lambda_7}{k^5}\beta_3-\frac{\lambda_8}{2k^3}\beta_4\right)h.$$

It is clear that if h is large enough, then $I_7(h)$ has the different sign with β_4. Namely, we can choose k_4 such that for $k\geqslant k_4$ the lemma holds. □

Lemma 4. *There exists a constant k_5 such that for any $k>k_5$, $I_7(h)$ at $h=H(P_4)$ has the same sign with β_4.*

Proof. Suppose that the curve $\{(x,y)\mid H(x,y)=H(P_4)\}$ intersects the x-axis at $(x_0,0)$, $(x_1,0)$ and $(\lambda,0)$, where $x_0<-1<x_1<0$, and intersects the line $y=k$ at (x_3,k) and (x_4,k), where $x_3<-1<\lambda<x_4$, see Fig. 4. Then

$$I_7\big(H(P_4)\big)=2\int\limits_{(x_0,0)}^{(\lambda,0)} y\big(\alpha_1+\alpha_2x+\alpha_3x^2+\alpha_4y^2\big)\,dx,$$

where the integral is taken from $(x_0,0)$ to $(\lambda,0)$ along the curve $\{(x,y)\mid H(x,y)=H(P_4)\}$ in the clockwise way.

Like in (12), we let $y^2-k^2=2\tilde y$. Then

$$\frac{I_7(H(P_4))}{2}=\int\limits_{(x_0,-\frac{k^2}{2})}^{(\lambda,-\frac{k^2}{2})}\sqrt{2\tilde y+k^2}\big(\alpha_1+\alpha_2x+\alpha_3x^2+\alpha_4\big(2\tilde y+k^2\big)\big)\,dx,$$

where the integral is taken from $(x_0,-\frac{k^2}{2})$ to $(\lambda,-\frac{k^2}{2})$ along the curve $\{(x,\tilde y)\mid \tilde y^2+F(x)=H(P_4)+\frac{1}{4}k^4\}$ in the clockwise way.

When $\tilde{y} > 0$, for convenience of exposition, we introduce two functions

$$G_1(\tilde{y}) = \begin{cases} \sqrt{1 + \frac{2\tilde{y}}{k^2}} - \sqrt{1 - \frac{2\tilde{y}}{k^2}} & \text{for } x \in (x_3, x_0) \cup (\lambda, x_4); \\ \sqrt{1 + \frac{2\tilde{y}}{k^2}} & \text{for } x \in (x_0, \lambda), \end{cases}$$

and

$$G_2(\tilde{y}) = \begin{cases} \sqrt{(1 + \frac{2\tilde{y}}{k^2})^3} - \sqrt{(1 - \frac{2\tilde{y}}{k^2})^3} & \text{for } x \in (x_3, x_0) \cup (\lambda, x_4); \\ \sqrt{(1 + \frac{2\tilde{y}}{k^2})^3} & \text{for } x \in (x_0, \lambda). \end{cases}$$

Now the Abelian integral $I_7(h)$ at $H(P_4)$ can be expressed as follows:

$$\begin{aligned} \frac{I_7(H(P_4))}{2} &= \int_{x_3}^{x_4} \bigl[G_1(\tilde{y})\bigl(k\alpha_1 + k\alpha_2 x + k\alpha_3 x^2\bigr) + k^3\alpha_4 G_2(\tilde{y})\bigr]\,dx \\ &= \frac{\beta_4}{2}\int_{x_3}^{x_4} \bigl(3G_1(\tilde{y}) - G_2(\tilde{y})\bigr)\,dx + \frac{1}{k^4}\int_{x_3}^{x_4} G_1(\tilde{y})\bigl(\beta_1 + \beta_2 x + \beta_3 x^2\bigr)\,dx. \end{aligned}$$

It is easy to prove that for any $t \in (0, 1)$,

$$3\bigl((1+t)^{\frac{1}{2}} - (1-t)^{\frac{1}{2}}\bigr) > (1+t)^{\frac{3}{2}} - (1-t)^{\frac{3}{2}}.$$

Thus if $x \in (x_3, x_0) \cup (\lambda, x_4)$, then $3G_1(\tilde{y}) - G_2(\tilde{y}) > 0$. When k is sufficiently large, for $x \in (x_0, \lambda)$, $\tilde{y}$ satisfies that

$$\tilde{y}^2 = -\frac{1}{12}\lambda^3(\lambda + 2) - F(x) + \frac{1}{4}k^4 \leqslant \frac{1.21}{4}k^4.$$

Therefore we have

$$\int_{x_3}^{x_4} (3G_1 - G_2)\,dx > \int_{x_0}^{\lambda} \left(1 + \frac{2\tilde{y}}{k^2}\right)^{\frac{1}{2}} \left(2 - \frac{2\tilde{y}}{k^2}\right) dx > 0.9(\lambda - x_0) > 0.$$

On the other hand,

$$F(x_3) = F(x_4) = -\frac{1}{12}\lambda^3(\lambda + 2) + \frac{1}{4}k^4.$$

Notice that $F(x)$ is a polynomial of degree 4. Thus if k is sufficiently big, then $|x_3|, |x_4| < 2k$, which implies that

$$\left|\int_{x_3}^{x_4} G_1(\tilde{y})\bigl(\beta_1 + \beta_2 x + \beta_3 x^2\bigr)\,dx\right| \leqslant 4k \cdot 3C \cdot (2k)^2 \cdot \max\bigl\{G_1(\tilde{y})\bigr\} \leqslant 96Ck^3,$$

where $C = \max\{|\beta_1|, |\beta_2|, |\beta_3|\}$. Obviously, if k is sufficiently big, then $I_7(H(P_4))$ has the same sign with β_4. □

Finally by combining the results of Lemmas 3 and 4, we prove the third statement of Proposition 1.

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