HOMEWORK 5 **Math 2921** *Ordinary Differential Equations II* - RUBIN - Spring '09

Problems 1-3 of this assignment are due in class on Wednesday, April 15th. Problems 4-6 of this assignment are due on Friday, April 24th.

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1. Apply averaging to the equations

$$\dot{x} = \epsilon(x - x^2)\sin^2 t$$

and

$$\dot{x} = \epsilon(x\sin^2 t - x^2/2)$$

to obtain all terms of $O(\epsilon)$ and $O(\epsilon^2)$. Compare and contrast the results for the two equations (i.e., there will be similarities and differences). Specifically, how does the existence of periodic orbits compare for the 2 equations?

2. Analyze the existence of periodic orbits, as a function of b, for the equation

$$\ddot{x} + x = \epsilon\dot{x}(-1 + bx^2 - x^4).$$

3. Consider the set $I = I_+ \cap I_-$ defined in class from intersections of vertical strips and horizontal strips under a Smale horseshoe map.

   a) List the periodic orbits within I with periods less than or equal to 4.

   b) Show that I contains orbits which are not asymptotically periodic and describe an example.

   c) Show that the set of heteroclinic orbits in I and the set of homoclinic orbits in I each form a dense set in I. To do this, define a metric d between two points $s, t \in I$ by taking the corresponding bi-infinite sequences in two symbols, call them $\mathbf{s} = \{..., s_{-1}, s_0, s_1, ...\}$, $\mathbf{t} = \{..., t_{-1}, t_0, t_1, ...\}$, and setting

$$d(s,t) = \sum_{i=-\infty}^{\infty} \frac{|s_i - t_i|}{2^{|i|}}.$$

   For any such $\mathbf{s}$, define the stable manifold $W^s(\mathbf{s})$ as the set of sequences with entries that agree with $\mathbf{s}$ for all entries to the left of some entry of $\mathbf{s}$, and define $W^u(\mathbf{s})$ analogously (replace left with right). There is a 1-1 correspondence between the heteroclinic orbits in I and the heteroclinic sequences, namely all sequences in $W^s(\mathbf{0}) \cap W^u(\mathbf{1})$ or in $W^s(\mathbf{1}) \cap W^u(\mathbf{0})$. There is also a 1-1 correspondence between the homoclinic orbits in I and the homoclinic sequences, namely all sequences in $W^s(\mathbf{0}) \cap W^u(\mathbf{0})$ or in $W^s(\mathbf{1}) \cap W^u(\mathbf{1})$. Establish the density of the set of heteroclinic sequences and of the set of homoclinic sequences in the family of bi-infinite sequences on two symbols.

4. Compute the Melnikov function for the equation

$$\ddot{x} - x + x^2 = \epsilon a\dot{x} + \epsilon\sin(\omega t)$$

and analyze where it has simple zeroes.

5. Consider the equation

$$\ddot{x} + \sin x = \epsilon(\beta - \alpha\dot{x}), \tag{1}$$

where $\alpha, \beta > 0$ and $0 < \epsilon \ll 1$.

a) Sketch the phase portrait for $\epsilon = 0$ and observe that there are two heteroclinic orbits (mod 2π), one above the x-axis and one below it. These can be denoted $q^{\pm}(t) = (x^{\pm}(t), \dot{x}^{\pm}(t))$, parameterized such that $x^{\pm}(0) = 0$.

b) Analogously to the Melnikov theorem from class, if we rewrite (1) as the first order system $\dot{y} = f(y) + \epsilon g(y)$, then zeroes of the Melnikov functions

$$M^{\pm} = \int_{-\infty}^{\infty} f(q^{\pm}(t)) \wedge g(q^{\pm}(t))\, dt$$

yield the existence of heteroclinic orbits to (1) for $\epsilon > 0$ sufficiently small. Find parameter values at which such orbits exist.

6. The nondimensionalized system of equations

$$\begin{aligned} \dot{y}_1 &= \epsilon v_1 \\ \dot{y}_2 &= \epsilon v_2 \\ \dot{v}_1 &= -v_1 + \sin y_1 \cos y_2 \\ \dot{v}_2 &= -v_2 + W - \cos y_1 \sin y_2 \end{aligned} \tag{2}$$

describes the motion of an aerosol particle, with center position $y(t)$ and velocity $v(t)$, in a vertically aligned cellular flow field, where W is the particle's settling velocity in still fluid and $\epsilon \ll 1$ is its inertial response time. For $\epsilon = 0$, this system has an attracting torus of critical points, and for $\epsilon > 0$ sufficiently small these perturb to an invariant manifold M_ϵ defined by $v_1 = \sin y_1 \cos y_2 + h(y_1, y_2, \epsilon)$, $v_2 = W - \cos y_1 \sin y_2 + k(y_1, y_2, \epsilon)$. Expand h, k in powers of ϵ and solve for the leading order ($O(\epsilon)$) term of each expansion. Use the full expansion to prove that on M_ϵ, each line $\{y_1 = m\pi\}$, for integer m, is invariant under the flow of equations (2).